Lesson - 1

ANALYSIS OF THE TIME SERIES

OBJECTIVES:

By the study of this lesson, you will be able to know the Meaning, Defintion, Objectives and Significance of the Analysis of Time series.

STRUCTURE:

1.1 MEANING OF TIME SERIES:

By the time series we mean a series of values of a variable the values of which vary according to the passage of time. In such type of variables the time factor plays an important role in affecting the variable to a market extent. The examples of such series may be cited as under.

Examples :

i) A series relating to consumption, production or prices of certain goods.

ii) A series relating to Purchase, Sales, Profits or Losses of a certain business concern.

iii) A series relating to agricultural or industrial production, investments, foreign exchange reserves, population, crimes, national income or imports and exports of a company.

iv) A series relating to bank deposits, bank clearings, prices of shares or dividend rates of a certain company.

v) A series relating to temperature, rainfall or yield of a particular area.

The values of the variables cited as above change very much due to the force or effect of time . A knowledge of such time effect over such variables is very much necessary and useful in predicting their future values and taking wise decisions as and when required. A critical appreciation or statistical analysis of such type of series is called " Analysis of Time series". The analysis of such type of series becomes indispensable in certain fields like business, economics, politics, sociology etc. where at average stage it is necessary to predict or forecast the values of certain variables to take the wise decisions and adopt the necessary plans to suceed in a particual pursuit.

1.2 DEFINITIONS :

Time series cited as above has been defined variously by various authors. Important of them are quoted as under.

1) According to Croxton and Cowden " A Time series consists of data arrayed chronologically ".

2) According to kenny and keeping " A set of data depending on the time is called time series ".

3) According to Ya-lun-chou, " A time series may be defined as a collection of readings belonging to different time periods of some economic variable or composite of variable such as production of steel, per-capita income, gross national products, price of tobacco or index of industrial production.

4) According to Wessel and Wellet " When quantitative data are arranged in the order of their occurance the resulting statistical series is called a time series ".

1.3 OBJECTIVES :

In the words of Hirsch, " The main objective of analysing time series is to understand interpret and evaluate changes in economic phenomena in the hope of more correctly anticipating the course of future events".

Thus the main objectives of analysing a time series may be noted as under.

1) To evaluate the post performance or occurance of a paritcular variable and

2) To forecast and predict the magnitude of a variable in furture so as to arrive at a desired conclusion for one's future course of action.

1.4 SIGNIFICANCE OF TIME SERIES :

A study of time series throws light on the furture economic behaviour of a certain variable. Forecasting with reasonable accuracy of the future economic conditions is an indispensable need for all successful planning. The analysis of time series helps to study the factors or forces that influence the changes in economic activities and predict the future variations in them with certian limitations.

The analysis of time series reflects the dynamic pace of movements of phenomenon over a period of time. It has an important and significant place in business and economics it is also helpful to sociologists, biologists, doctors and workers. It determines the pattern of data collected over a period of time, helps in understanding the past behaviour of data and guides in predicting the future behaviour of data and guides in predicting the future behaviour of the same. It is also of much importance in evaluating the current performance of the data and in comparing the actual performance with the predicted performance.

1.5 SUMMARY :

A Time series is a series of numerical data which have been recorded at different intervals of time. It is a record of changes in variables over a period of time. In this series, the observations are arranged in a chronological order.

1.6 QUESTIONS :

1. Explain the meaning of Time series with examples.
2. Define Time series.
3. What are the objectives of Time series Analysis ?
4. What is the significance of Time series ?
5. What are the uses of Time series ?

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Lesson - 2

COMPONENTS OF TIME SERIES

OBJECTIVES:

By the study of this lesson, you will be able to understand the meaning of the components of time series. Various components of time series (Secular, cyclical, seasonal and Random fluctuations) and the various adjustments to be made in the data before making the analysis of Time series.

STRUCTURE:

2.1 Introduction

2.2 Secular Fluctuations

2.3 Cyclical Fluctuations

2.4 Seasonal Fluctuations

2.5 Random Fluctuations

2.6 Preparation of data before analysis

2.7 Summary

2.8 Questions

2.1 INTODUCTION :

The changes or variations in the values of any Phenomenon over a period of time are due to a number of forces or factors or componenets and their cumulative effects. The forces or factors, due to which the values of Phenomenon over a period of time change, are called the "Components of Time Series ".

If the values of the Phenominon are plotted on a graph and the plotted points fall more or less in the pattern around a straight line, it is called a " Linear Trend ". If the values of the phenomenon do not fall in the pattern of a straight line, it is called a " Non - Linear" or " Curvilinear Trend " . In a linear trend, the values increases or decrease by a constant absolute amount. In a non - linear trend, the values have no particular trend in their changes.

Thus the values of phenomenon or variable change due to the interaction of a number of forces or components or factors which are complex and dynamic in nature. The components of Time series are classified as under.

COMPONENTS OF TIME SERIES

Long term	Medium term	Short term	Residual
Secular	Cyclical	Seasonal	Random

2.2 SECULAR FLUCTUATIONS :

The values of variables tend to increase or decrease (not both) over a period of 6 to 7 years. Though there are upward and downward changes in the trend, we find in general, only one trend over a period of time i.e, eigther and upward trend or a downward trend or sometime constant trend. the fluctuations indicate a simple or secular trend, which depicts the smooth regular and slow movements in the changes in one direction.

The term " long term " is a relative term, which cannot be defined precisely. It is interpeted with reference to the nature of data and changes in the technology, institutions and culture. There will be an overall general tendency in the phenomenon. It is all due to the tastes, habits and customs of the people. It may also be due to new techniques, new products and new substitutes that the changes take place. Steady growth in the population, gradual decline in the death rate and slow increase in the demand for electronic goods are the examples involving secular forces or components. Secular trend or changes is the basic tendency of time series with which a statistician is more concerned.

2.3 CYCLICAL FLUCTUATIONS :

The values of variable change in a rythmic pattern over a period of time from 2 to 6 years extending even upto 8 years. There are oscillatory movements in the economic activities, which are complex, less regular and less uniform. They are the outcome of the business cycles - Boom Period (Prosperity) Recession (Downward), Depression (Reaching a rock bottom), and Recovery (Picking up again) and agian Boom period. They generally relate to the economic activities. The fluctuations are not necessarily regular, uniform or even. They man or man not follow a similar pattern.

In practice, for the purpose of effective economic offsetting action, anticipation of cyclical fluctuations is very essential. There are waves like movements in economic activities.

2.4 SEASONAL FLUCTUATIONS :

The values of variables change in a regular, periodic and rhytmic order according to season covering a period of time not exceeding 12 months. Every year the fluctuations are noticed regulary influencing the economic activities. Most of the time series are influenced by the seasonal forces. They can be predicted accurately.

The seasonal factors or forces may be " natuaral components" Weather condition, climatic changes, rainfalls and sun-lights or man made conventions - habits, customs, fashions, marriages, traditions and festivals. All these components occur at regular intervals during the year. A study of seasonal fluctuations is of utmost importance to the business man in framing the policies.

2.5 RANDOM FLUCTUATIONS :

The values of variables change suddenly or unexpectedly. The influencing factors are unforseeen and unpredictable. They are generally mixed up with the seasonal and cyclical fluctuations. They are irregular, erratic and accidental, they donot have repetitive tendency. They donot show any definite pattern or period. They are caused by non - recurring factors like strike, lockouts, floods, famines, wars, earthquakes, epedemics, revolutions and storms.

Normally the random components play their role for a few days or months but their effects are so intense that they may give rise to new cyclical movements which are abnormal in nature. It is very difficult to study them exclusively and isolate them.

2.6 PREPARATION OF THE DATA BEFORE ANALYSIS :

Before starting with the actual analysis of a time series, it is necessary to prepare the data for the purpose. For this, certain adjustments are to be made to bring about homogeneity in the data in certain respects as follows -

i) **Adjustment for Calender variation :** It is a fact that different months have different number of days and so the monthly results will naturally show different figures even though the daily results might be the same throughout each of the months. Thus to bring out a homogeneity in the monthly results we have to devide the monthly total by the number of days (or working days) in the respective months and multiply the same by the average number of days in a month i.e. 30 approx. This way we will get the adjusted monthly values of the variable during a year.

ii) **Adjustment for Population variation :** Certian variables like national income, national expenditure, national production, national consumption etc. vary according to the increase or decrease in the size of population. It we go by the total figures of such variables without having regard to the increase or decrease in the size of population we will be making a fallacious conclusion. In such type of data the per-capita figures should be substituted for the total figures to arrive at the appropriate results. For this the total figure of any such variable is to be devided by the number of population during the year concerned.

iii) **Adjustment for Price variation :** There are certain variables like Income, Expenditure values of sales etc, which are very much affected by the changes in the price level. Inorder to make an appropriate analysis of such data their current values should be adjusted in the light of price indices as under.

$$\text{Real value} = \text{Current value} \times \frac{\text{Base price index}}{\text{Current price index}}$$

The above process of converting the current values into real values is called the process of deflation.

iv) Adjustment for comparability : Analysis of Time series reveals many valuable aspects and differences between two or more related variabels by the process of comparision. Comparsion is made possible and meaningful if the meaning of the various terms and units used in the concerned variables remain the same. But in actual practice there appear some differences in the meaning of certain vital terms and units. For example, in certain years, the wage might have been taken as money wage while in certain other years it might have been taken as real wage. In such cases the figures will not be comparable with each other. Therefore uniformity in the concept of such terms must be brought about first before making further analysis on them.

2.7 SUMMARY :

The main components of Time series are classified into 4 types. They are Secular, Cyclical, Seasonal and Random.

2.8 QUESTIONS :

1. Explain the meaning of the components of Time series.
2. What are the different components of Economic Time Series ?
3. What do you understand by Secular fluctuations ?
4. What is meant by Cyclical fluctuations ?
5. Critically examine the Seasonal fluctuations ?
6. What do you understand by Random fluctuations ?
7. Name the methods of determining seasonal index.
8. How would you statistically eliminate the influence of seasonal and cyclical factors on the long period movement of any time series ?
9. What is meant by seasonal dictations of a time series.
10. Clearly explain the adjustments to be made in the data before making the time series analysis.

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Lesson - 3

TIME SERIES -

COMPUTATION OF TREND VALUES - I

OBJECTIVES:

By the study of this lesson, you will be able to understand the calculation of Trend values under Graphic or Free hand Curve menthod and Semi- Averages method with suitable graphs and Examples with solutions.

STRUCTURE:

3.1 INTODUCTION :

Generally the collected data will be in the vast and complex form. It will be the composition of several forces which may be pulling together the trend in different directions. The problem will be solved by statisticans by decomposing the time series to understand the historical past and to give insight into the future. There are four methods commonly used in measuring the trend values.

i) Graphic method or Free hand curve method.

ii) Semi - Averages method.

ii) Moving - Averages method.

ii) Least - Squares method.

3.2 GRAPHIC OR FREE HAND CURVE METHOD :

A trend is determined by just inspecting the plotted points on a graph sheet. Observe the up and down movements of the points. Smooth out the irregularities by drawing a free hand curve or line through the scatter points. The curve so drawn would give a general notion of the direction of the change. Such a free hand smoothed curve eliminates the short time swings and shows the long period general tendency of the changes in the data.

Drawing a smooth freehand curve requires a personal skill and judgement. The drawn curve should pass through the plotted points in such a manner that the variations in one direction are approximately equal to the variations in other direction. Different persons, how ever, draw different curves at different directions with different slopes and different styles. This may lead to different conclusions. To overcome these limitations, we can use the Semi- Averages method of measuring the trend.

3.3 SEMI - AVERAGES METHOD :

Under this method there are two types of calculating the semi - Averages.

a) When the given items are 'odd ' in number

b) When the given items are 'even' in number

a) When the given items are ' odd ' in number :

If the given items are ' odd ' in number, the middle item is to be left out and the averages must be calculated for the upper part and lower part seperately.

3.4 EXAMPLES :

Examples 1 :

The price of a product in 7 years was as under .

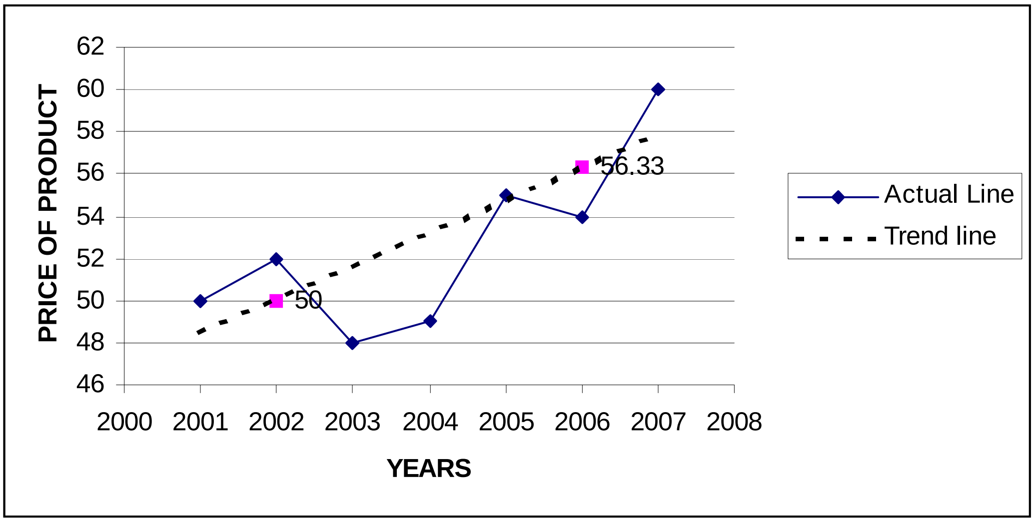
Year	-	2001	2002	2003	2004	2005	2006	2007
Price (in Rs.)	-	50	52	48	49	55	54	60

Solution :

Year	**x**		**Total**	**Semi Averages**
2001	50			
2002	52	Part I	150 ÷ 3	50.00
2003	48			
2004	49	Left out		
2005	55			
2006	54	Part II	169 ÷ 3	56.33
2007	60			

The total of the values in each part is devided by the number of values of that part. The result values will be the Mean values of each part. Each average is then centered in the period of time from which we have to plot the two points on the graph i.e. " Two Average values are plotted". Then both these two points are joined by a straight line which gives the simple Semi-Average trend.

GRAPH SHOWING PRICE OF A PRODUCT WITH TREND LINE



b) When the given items are 'even' in number :

Examples 2 :

Using the method of Simple semi - Average determine the trend of the following data.

Year	-	2000	2001	2002	2003	2004	2005	2006	2007
Production of sugar (in '000 tons)	-	32	28	40	36	24	46	56	50

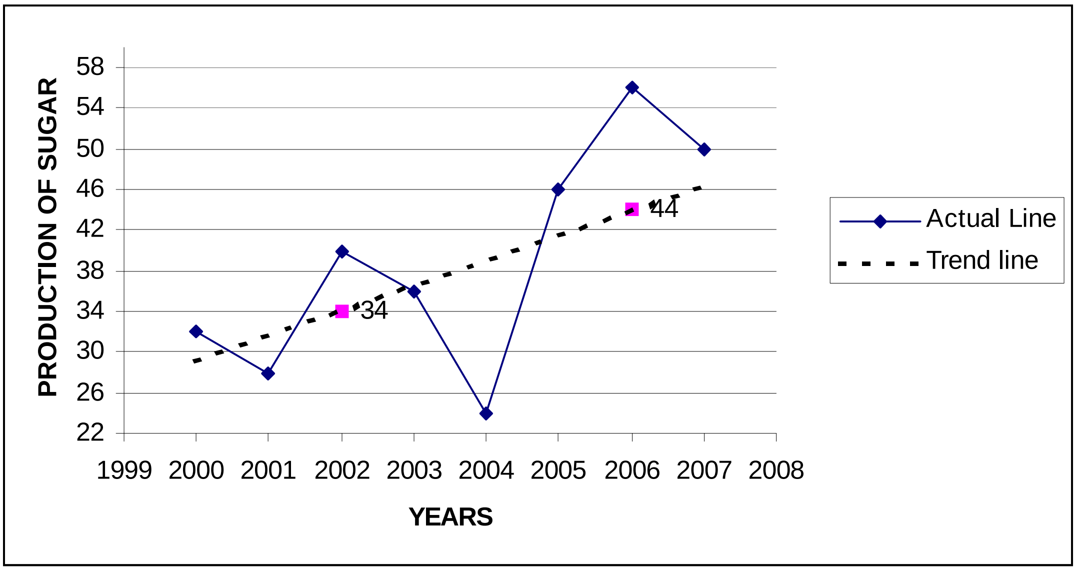
Solution :

The number of observations are 'even' i.e 8. The series must be devided into two equal parts (first 4 and last 4) 1st part will be from 2000 to 2003 and the last part will be from 2004-2007. Averages must be calculated for the two parts seperately.

Year	x	
2000	32	
2001	28	
2002	40	136 ÷ 4 = 34
2003	36	
		No middle item
2004	24	
2005	46	
2006	56	176 ÷ 4 = 44
2007	50	

The value 34 is plotted against the middle of the first four years i.e. 2000 to 2003. The value 44 is plotted against the middle of the last four years i.e 2004 to 2007. Then both the points are joined by a straight line as under.

GROUP SHOWING SUGAR PRODUCTION WITH TREND LINE



3.5 MERITS OF SEMI - AVERAGES METHOD :

1) This method is simple to follow and compute.

2) This method is an objective one, there will not be variations in the answers, if trend values are calculated by two or more person. It is not depending upon the personal judgement like freehand smoothed curve drawing.

3.6 LIMITATIONS OF SEMI - AVERAGE METHOD :

1) The method is based on the assumption that there is a linear trend which may not be true in all the cases.

2) Under this method the trend is greatly affected by the extreme values as it is based on the arithmetic mean.

3) This method is not suitable when time period represented by average is small.

3.7 SUMMARY :

The vast and complex data will be decomposed and trend is calculated under the Free hand curve method and Semi- Averages method.

3.8 QUESTIONS :

1. What is meant by Trend ? How it is computed ?
2. Explain the method of Calculation of trend under Graphic or Free hand curve method.
3. Explain the meaning of Semi - Averages.
4. How the Trend is calculated under Semi - Averages method ?

3.9 EXERCISES :

1. Compute the trend value under the Semi- Averages method.

Year	-	2001	2002	2003	2004	2005	2006	2007
Production (in '000'tons)	-	40	45	43	48	52	47	44

2. Calculate the trend values and draw a graph under Semi- Averages method.

Year	-	1996	1997	1998	1999	2000	2001	2002	2003	2004
Price of a product in Rs	-	74	76	80	78	73	81	83	77	83

3. The production in a factory during last 8 years was as under. Compute the Trend values and show the data and trend values on a graph paper.

Year	-	2000	2001	2002	2003	2004	2005	2006	2007
Production (in '000'tons)	-	275	290	310	425	470	540	600	620

4. The sales of a company in lakhs of rupees were as under. Calculate the Semi - Average and show the data and the Semi - Averages in a Graph.

Year -	2002	2003	2004	2005	2006	2007
Sales -	75	92	100	80	84	110

5. The exports from India during the last 7 years were as under. Calculate Trend values under the Semi - Averages method and show the data and trend on a graph.

Year -	2001	2002	2003	2004	2005	2006	2007
Exports (in crores) -	470	450	540	500	530	670	740

6. The Imports of India during the last 8 years were as under. Calculate the trend values under the Semi - Averages method.

Year -	2000	2001	2002	2003	2004	2005	2006	2007
Imports (in crores) -	79	85	115	112	113	99	87	103

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Lesson - 4

TIME SERIES -

COMPUTATION OF TREND VALUES - II

(Moving Averages Method)

OBJECTIVES:

By the study of this chapter you will be able understand the meaning of moving Averages method, computation of Trend values under Moving Averages method with suitable graphs examples with solutions and merits & limitations of moving averages method.

STRUCTURE:

4.1 Introduction

4.2 Examples

4.3 Merits of Moving Averages

4.4 Limitations of Moving Averages

4.5 Seasonal Index

4.6 Summary

4.7 Questions

4.8 Exercises

4.1 INTODUCTION :

Moving averages are calculated for 3years or 4years or 5years or 6years or 7 years or 8years or 9 years and so on depending upon the size of the items.

The method of calculation of these averages will be as under.

3 years Moving Averages : $\frac{A+B+C}{3}$; $\frac{B+C+D}{3}$; $\frac{C+D+E}{3}$; $\frac{D+E+F}{3}$; $\frac{E+F+G}{3}$

4 years Moving Averages : $\frac{A+B+C+D}{4}$; $\frac{B+C+D+E}{4}$; $\frac{C+D+E+F}{4}$; $\frac{D+E+F+G}{4}$

5 years Moving Averages : $\frac{A+B+C+D+E}{5}$; $\frac{B+C+D+E+F}{5}$; $\frac{C+D+E+F+G}{5}$; $\frac{D+E+F+G+H}{5}$

6 years Moving Averages : $\frac{A+B+C+D+E+F}{6}$; $\frac{B+C+D+E+F+G}{6}$; $\frac{C+D+E+F+G+H}{6}$

Under this method, the trend values can be obtained by employing arithmetic means of the series except at the two ends of the series. The procedure of averaging the 3items or 4 items or different items simplifies the analysis and removes variations in the values for a period concerned. The moving averages may form a straight line trend or a curve.

The calculation of moving averages in case of odd number of years is very simple as the average can be placed at the centre of the period. In case of even number of years, the averages fall in between the two years or periods. This again requires to be adjusted in such a way that the averages shall be placed against the respective years. So we can have the arithmetic mean of the two averages that shifts the values to the respective years. This process of shifting the figures is "Centering " the averages.

In the case of 3 years moving averages, take the total of first three items (A, B and C) and put the total and the average against the 2nd item (against B). Then take the total of next three items. (B, C and D) and put the total and average against the third (B) item. This procedure is continued till the end. The first and last items remain without the average figures.

In the case of 4 years moving averages, take the totals of first four items (ABC&D) and put the total and average in between 2nd and 3rd items (between B&C). Then continuing the next process, put the total and averages in between 3rd & 4th items (between C&D). In this way we have the 4 years moving averages which are laying between the two periods. This needs the procedure of shifting or centering the averages. If we simply obtain the means of the two averages, the new averages will be centered exactly against the periods mentioned.

The moving averages can be neatly presented in the form of a graph. If a series contains no fluctuations the values plotted on a graph give a straight line trend to some extent.

4.2 EXAMPLES :

Examples 1 :

From the following data. Calculate the 3years moving averages.

Year	-	1996,	97,	98,	99,	2000	01	02	03	04	05	06	07
Sales (in lakhs)	-	32	38	42	44	46	50	48	44	52	52	54	52

Solution : Computation of Trend values (3years Moving Averages)

Year	**Sales**	**Total of 3years**	**3years M.A.**
1996	32		
97	38	112	37.33
98	42	124	41.33
99	44	132	44.00
2000	46	140	46.07
01	50	144	48.00
02	48	142	47.33
03	44	144	48.00
04	52	148	49.33
05	52	158	52.67
06	54	158	52.67
07	52		

Examples 2 :

Find out trend values through 5years moving averages method from the following data.

Year	- 1991,	92,	93,	94,	95,	96,	97,	98,	99,	2000,	01,	02,
Exports (Rs. in crores) -	64	76	84	88	92	100	96	88	104	104	108	104

Year	**Exports**	**Total of 5years**	**5years M.A.**
1991	64		
92	76		
93	84	404	80.8
94	88	440	88.0
95	92	460	92.0
96	100	464	92.8
97	96	480	96.0
98	88	492	98.4
99	104	500	100.0
2000	104	508	101.6
01	108		
02	104		

Examples 3 :

From the following data. Calculate the 4years moving averages and centralised average.

Year	-	1996,	97,	98,	99,	2000	01	02	03	04	05	06	07
Production (in MT)	-	50	54	57	56	55	58	59	60	63	65	68	70

Solution : Computation of Trend values (4years Moving Averages)

Year	**Production**	**Total of 4years**	**4years M.A.**	**Total of 2yearsM.A**	**Centralised Average**
1996	50				
97	54				
		217	54.25		
				109.75	54.875
98	57				
		222	55.50		
				112.00	56.000
99	56				
		226	56.50		
				113.50	56.75
2000	55				
		228	57.00		
				115.00	57.50
01	58				
		232	58.00		
				118.00	59.00
02	59				
		240	60.00		
				121.75	60.875
03	60				
		247	61.75		
				125.75	62.875
04	63				
		256	64.00		
				130.50	65.25
05	65				
		266	66.50		
06	68				
07	70				

Examples 4 :

From the following data, Calculate the trend values under 4years Moving Average method. Also find out the centralised Average.

Year	- 1994,	95	96,	97,	98,	99,	2000	01	02	03	04	05
Exports (Rs in croses) -	70	80	78	82	88	104	109	100	99	112	113	123

Solution : Computation of Trend values (4years Moving Averages)

Year	Production	Total of 4years	4years M.A.	Total of 2years average	Centralised Average
1994	70				
95	80				
		310	77.50		
				159.56	79.75
96	78				
		328	82.00		
				170.00	85.00
97	82				
		352	88.00		
				183.75	91.875
98	88				
		383	95.75		
				196.00	98.00
99	104				
		401	100.25		
				203.25	101.625
2000	109				
		412	103.00		
				208.00	104.00
01	100				
		420	105.00		
				211.00	105.50
02	99				
		424	106.00		
				217.75	108.875
03	112				
		447	11.75		
04	113				
05	123				

Examples 5 :

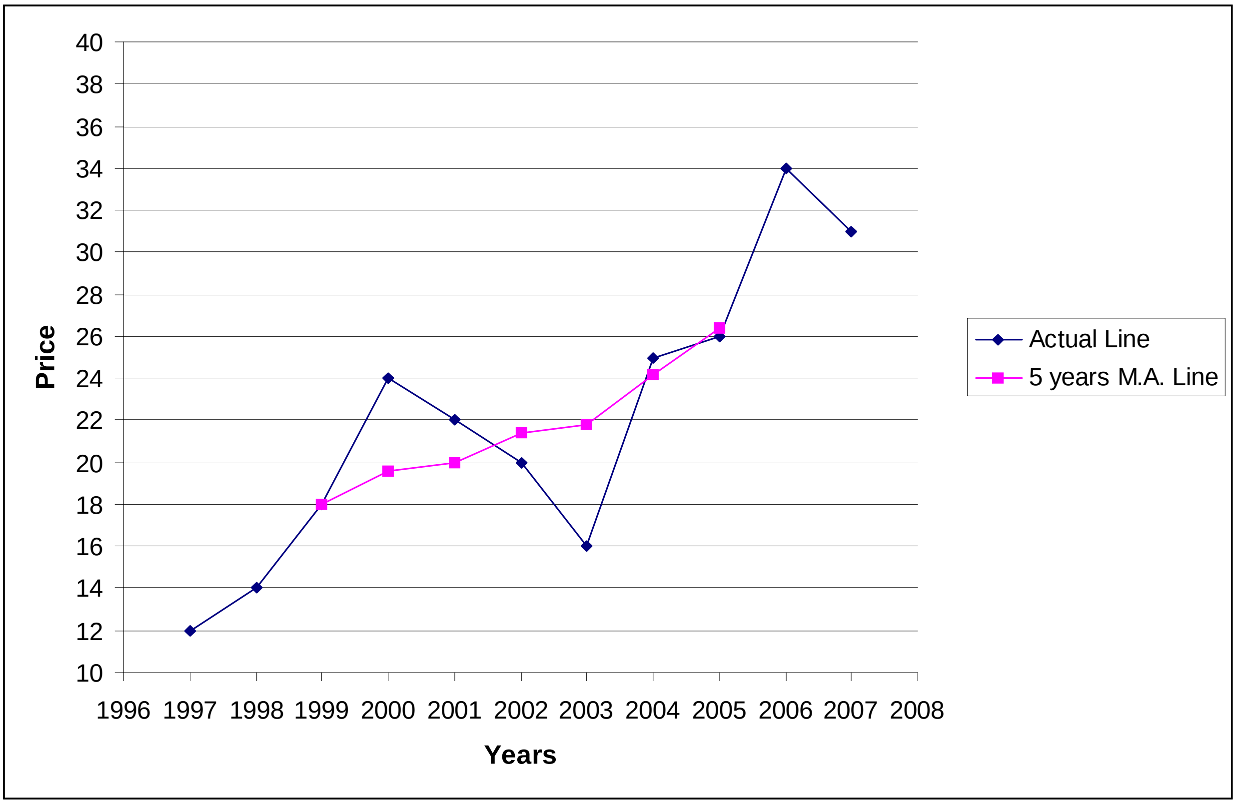
Calculate the trend values by five yearly moving average method and plot the same on a graph from the following.

Year	-1997,	98,	99,	2000	01	02	03	04	05	06	07
Price (in Rs.) -	12	14	18	24	22	20	16	25	26	34	31

Solution : Computation of Trend values (5years Moving Averages)

Year	Sales	Total of 5years	5years M.A.
1997	12		
98	14		
99	18	90	18.0
2000	24	98	19.6
01	22	100	20.0
02	20	107	21.4
03	16	109	21.8
04	25	121	24.2
05	26	132	26.4
06	34		
07	31		

Graph showing Price of product in IIYears and trend values of 5 Years.



4.3 MERITS OF MOVING AVERAGE METHOD :

1) It is very simple in calculation. There are no higher degree mathematical calculations.

2) It is objective method, there will not be variations in the answers, if trend values are calculated by two or more persons. It is not depending upon the personal judgement like free hand smoothed curve drawing.

3) It considers all the values in the series. The extreme values are also included in the process of calculating averages.

4) It not only determines the trend values but also reflects light on the second, cyclical and irregular variation.

5) It tries to eliminate oscillatory movements in the time series by selecting a proper period for averages.

6) It is a flexible method as we can add a few more items to the given data without disturbing the trend values already calculated.

4.4 LIMITATIONS :

1) This Method fails to provide trend values for the early part and last part of the series. We have to leaveout the trend values for both the end of the series.

2) It does not establish functional relationship between the period(x) and the value of variable (y)

3) The averages obtained in case of non - linear trend are biased and the lie either above or below the true sweep of the data.

4) When the period selected is not appropriate the moving averages fail to represent a true picture of the real trend.

Seasonal Index :

Examples 6 :

From the following data find out seasonal index

Year	-	2003	2004	2005	2006	2007
I Quarter	-	72	76	74	76	78
II ,,	-	68	70	66	74	74
III ,,	-	80	82	84	84	86
IV ,,	-	70	74	80	78	82

Solution :

Year	**I Quarter**	**II Quarter**	**III Quarter**	**IV Quarter**
2003	72	68	80	70
2004	76	70	82	74
2005	74	66	84	80
2006	76	74	84	78
2007	78	74	86	82
	$\frac{376}{5}$	$\frac{352}{5}$	$\frac{416}{5}$	$\frac{384}{5}$
=	75.2	70.4	83.2	76.8

$$\text{Annual average} = \frac{75.2 + 70.4 + 83.2 + 76.8}{4} = \frac{305.6}{4} = 76.4$$

$$\text{Seasonal average or Index} = \frac{\text{Quarter Average}}{\text{Annual Average}} \times 100$$

$$\text{I Quarter} = \frac{75.2}{76.4} \times 100 = 98.43\%$$

II ,, $= \frac{70.4}{76.4} \times 100 = 92.15\%$

III ,, $= \frac{83.2}{76.4} \times 100 = 108.9\%$

IV ,, $= \frac{76.8}{76.4} \times 100 = 100.52\%$

4.6 SUMMARY :

Method of moving averages just reduces the irregular fluctuation in the values over a period of time. As the period covered is larger, the degree of reduction in irregularities will be greater. It reproduces the linear trend, if the period covered is long. How ever too long a period of time, some times, is likely to lead further fluctuations in the averaged values. That is why the period covered should neither be too long nor too short.

4.7 QUESTIONS :

1. Explain the meaning of Moving Averages method.
2. What are the merits of Moving Averages method.
3. What are the limitations of Moving Averages method.

4.8 EXERCISES :

1. Take 3 yearly period of moving average and determine the trend line. Show the data on a graph.

Year	-	1	2	3	4	5	6	7	8	9	10	11	12
Production	-	14	17	22	28	26	18	29	24	25	29	30	23

2. Calculate 5 yearly moving average to determine the trend value. Plot the data on a graph paper.

Year	-	1996	97	98	99	2000	01	02	03	04	05	06	07
Sales	-	50	52	55	53	58	59	62	67	64	63	58	69

3. Calculate the trend values under 5 yearly moving average method and plot the data on a graph paper.

Year	-	1991	92	93	94	95	96	97	98	99	2000	01	02	03
Prices.	-	21	23	22	27	25	29	21	18	20	25	28	33	35

4. Calculate the trend values under 5 yearly moving average method

Year -	1995,	96	97,	98,	99,	2000	01	02	03	04	05	06
Imports -	150	172	164	165	171	182	156	153	170	168	169	161

5. Calculate the 4 yearly moving average from the following data and show it on a graph paper

Year -	1996	97	98	99	2000	01	02	03	04	05	06	07
Exports -	245	275	280	296	252	258	276	304	315	364	301	313

6. Calculate the 4 yearly and 5 yearly Moving averages.

Year -	1992	93	94	95	96	97	98	99	2000	01	02	03
Consumption-	75	76	74	80	78	79	71	77	73	72	81	83

7. From the following data calculate seasonal index.

Year	I Quarter	II Quarter	III Quarter	IV Quarter
2004	3.7	4.3	3.3	3.5
2005	3.7	3.9	3.6	3.6
2006	4.0	4.1	3.3	3.1
2007	3.3	4.4	4.0	4.0

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Lesson - 5

TIME SERIES -

COMPUTATION OF TREND VALUES - III

(Least Squares Method)

OBJECTIVES:

By the study of this lesson you will be able to understand :- What is least squares method? how to compute it ? how to compute the trend values ? and how to findout the future and past values ? with suitable example and graphs.

STRUCTURE:

5.1 INTODUCTION :

This is the most accurate method of finding the trend values with the help of mathematical technique, which give us a straight line trend. It is a line from which actual values deviate on either sides. The sum of the deviations taken from the arithmetic mean will be zero. Consequently the sum of squares of deviations will be least as compared to the other alternatives. That is why this method is called the " Method of Least squares ".

A mathematical functional relationship is established between the two variables in the least squares method of finding the trend line. It satifies the following two conditions.

1) the sum of deviations of the ' Y ' (actual values) from the ' Yc' (computed values) is equal to zero.

Y = Actual value

Yc= Trend values

$\in$ (Y-Yc) = 0

2) The sum of the squares of x values is the minimum.

In time series the variable 'dx' can be measured from any point of the origin period first year or starting point. It is always better to select the point of the origin in the middle of the 'x' series (i.e. $\in dx = 0$). This brings balance between the negative and positive values. The period indicating the higher values can be reduced to the minimum symbolically.

The following procedure must be followed to find out the trend values under Least Squares method.

Yc = a +b dx

Yc= Trend values

$$a = \text{Arithmetic mean} = \frac{\in y}{N}$$

$$b = \frac{\in dxy}{\in dx^2}$$

dx = deviations taken from the assumed mean

Note : Assumed mean must be selected or taken from the years. The middle year shall be taken on assumed mean, so that $\in dx = 0$

$$\in (Y-Yc) = 0$$

5.2 EXAMPLES :

Examples 1 :

Compute the trend values (Yc) under least squares method from the following data. Estimate the sales for 2010.

Year	-	2003	04	05	06	07
Sales (Rs in crores)	-	70	74	80	86	90

Solution : Computation of Trend values

Year x	Sales Y	dx	dx^2	dxy	Yc
2003	70	-2	4	-140	69.6
2004	74	-1	1	-74	74.8
2005	80	0	0	0	80.0
2006	86	+1	1	+86	85.2
2007	90	+2	4	+180	90.4
N=5	400	0	10	+52	

$$a = \frac{\in y}{N} = \frac{400}{5} = 80$$

$$b = \frac{\in dxy}{\in dx^2} = \frac{52}{10} = 5.2$$

Yc = a +b dx

2003 = 80+5.2 (-2)

= 80 -10.4 = 69.6

2004 = 80 + 5.2 (-1)

= 80 -5.2 = 74.8

2005 = 80 +5.2(0) = 80.0

2006 = 80 + 5.2(1) = 85.2

2007 = 80 + 5.2(2)

= 80+10.4 = 90.4

Estimate for 2010 :

Yc = a + bdx

a= 80

b = 5.2

dx for 2007 = +2

2008 = +3

2009 = +4

2010 = +5

= 80 + 5.2 (5)

= 80 + 26.0

= 106

Mouthly increase in sales :

Yearly increase in sales = b = 5.2

$$\text{Monthly increase} = \frac{5.2}{12}$$

= 0.43

Examples 2 :

Find out trend values under least squares method from the following data. Plot the data on a graph paper

Year	-	2001	02	03	04	05	06	07
Profits (in lakhs)	-	60	72	75	65	80	85	95

Estimate the profits for 2009 and 1998

Solution : Computation of Trend values

x	**Y**	**dx**	**dx²**	**dxy**	**Yc**
2001	60	-3	9	-180	61.42
2002	72	-2	4	-144	66.28
2003	75	-1	1	-75	71.14
2004	65	0	0	0	76.00
2005	80	+1	+1	+80	80.86
2006	85	+2	4	+170	85.72
2007	95	+3	9	+285	90.58
N=7	532	0	28	+136	

$$a = \frac{\in y}{N} = \frac{532}{7} = 76$$

$$b = \frac{\in dxy}{\in dx^2} = \frac{136}{28} = 4.86$$

Yc = a +b dx

2001 = 76 + 4.86 (-3)

= 76 - 14.58 = 61.42

2002 = 76 + 4.86 (-2)

= 76 - 9.72 = 66.28

2003 = 76 + 4.86 (-1) = 71.14

2004 = 76 + 4.86 (0) = 76.00

2005 = 76 + 4.86 (1) = 80.86

2006 = 76 + 4.86(2)

= 76 + 9.72 = 85.72

2007 = 76 + 4.86(3)

= 76 + 14.58 = 90.58

Estimate for 2009 :

Yc = a + bdx

a= 76

b = 4.86

dx for 2006 = +2

2007 = +3

2008 = +4

2009 = +5

= 76 + 4.86 (5)

= 76 + 24.30

= 100.30

Estimate for 1998 :

Yc = a + b dx

a= 76

b = 4.86

dx for 2001 = -3

2000 = -4

1999 = -5

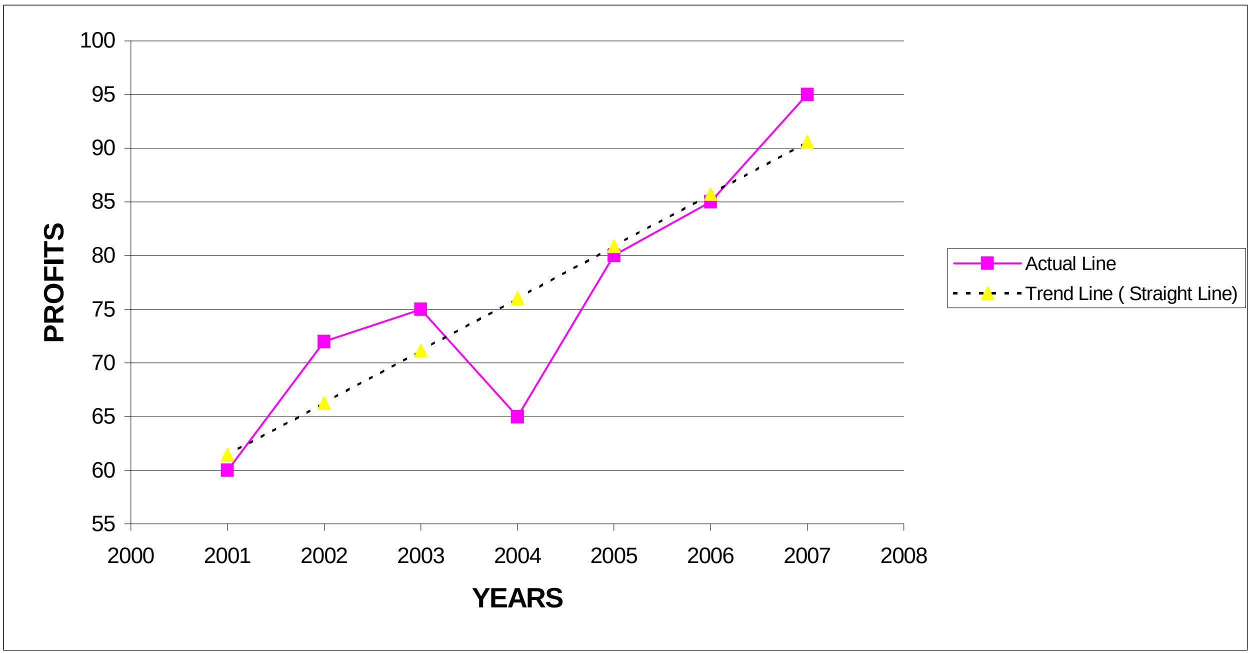
1998 = -6

= 76 + 4.86 (-6)

= 76 - 29.16

= 46.84

Comutation of Trend Values



Examples 3 :

Find out straight line trend under least squares method from the following data. Estimate the value for the year 2012

Year	-	2002	03	04	05	06	07
Values	-	83	92	71	90	169	191

Draw suitable graph by incorporating the above data and trend values.

Solution : Computation of Trend values

x	Y	dx	dx²	dxy	Yc
2002	83	-2.5	6.25	-207.5	59.48
2003	92	-1.5	2.25	-138.0	82.15
2004	71	-0.5	0.25	-35.5	104.72
		0	0	0	
2005	90	+0.5	0.25	+45.0	127.29
2006	169	+1.5	2.25	+253.5	149.86
2007	191	+2.5	6.25	+477.5	172.43
N=7	696	0	17.5	395.0	

$$a = \frac{\in y}{N} = \frac{696}{6} = 116$$

$$b = \frac{\in dxy}{\in dx^2} = \frac{395}{17.5} = 22.57$$

Yc = a +b dx

2002 = 116 + 22.57 (-2.5)

= 116 - 56.425 = 59.48

2003 = 116 + 22.57 (-1.5)

= 116 - 33.855 = 82.15

2004 = 116 + 22.57 (- 0.5)

= 116 - 11.285 = 104.72

2005 = 116 + 22.57 (0.5)

= 116 + 11.285 = 127.29

2006 = 116 + 22.57 (1.5)

= 116 + 33.855 = 149.86

2007 = 116 + 22.57 (2.5)

= 116 + 56.43 = 172.43

Estimate for 2012 :

a= 116

b = 22.57

dx for 2007 = 2.5

2008 = 3.5

2009 = 4.5

2010 = 5.5

2011 = 6.5

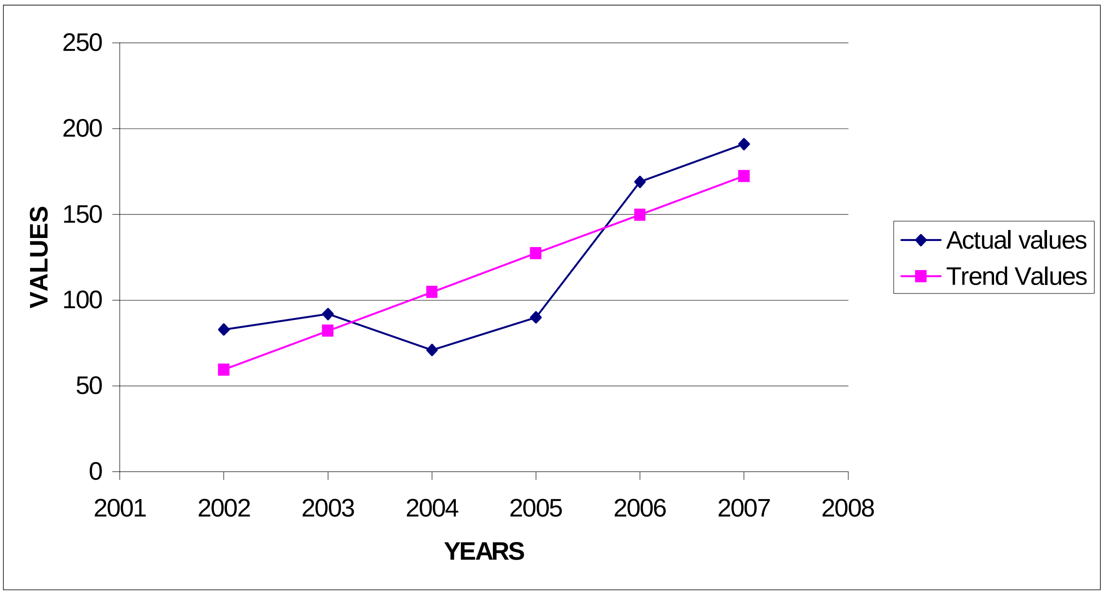
2012 = 7.5

= 116 + 22.57 (7.5)

= 116 + 169.28

= 285.28

Graphical Representation of Trend Values



Examples 4 :

Find out straight line trend equation under least squares method. Estimate the earnings for the year 2009.

Year	-	2000	01	02	03	04	05	06	07
Income	-	38	40	65	72	69	60	87	95

Solution : Computation of Trend values

x	Y	dx	dx²	dxy	Yc
2000	38	-7	49	-266	40.06
2001	40	-5	25	-200	47.40
2002	65	-3	9	-195	54.74
2003	72	-1	1	-72	62.08
		0	0	0	
2004	69	+1	1	+69	69.42
2005	60	+3	9	+180	76.76
2006	87	+5	25	+435	84.10
2007	95	+7	49	-665	91.44
N=8	526	0	168	616	

$a = \frac{\in y}{N} = \frac{526}{8} = 65.75$

$b = \frac{\in dxy}{\in dx^2} = \frac{616}{168} = 3.67$

Yc = a +b dx

2000 = 65.75 + 3.67 (-7)

= 65.75 - 25.69 = 40.06

2001 = 65.75 + 3.67 (-5)

= 65.75 - 18.35 = 47.4

2002 = 65.75 + 3.67 (-3)

= 65.75 - 11.01 = 54.74

2003 = 65.75 + 3.67 (-1)

= 65.75 - 3.67 = 62.08

2004 = 65.75 + 3.67 (1)

= 65.75 + 3.67 = 69.42

2005 = 65.75 + 3.67 (3)

= 65.75 + 11.01 = 76.76

2006 = 65.75 + 3.67 (5)

= 65.75 + 18.35 = 84.10

2007 = 65.75 + 3.67 (7)

= 65.75 + 25.69 = 91.44

Estimate for 2009 :

Yc = a + bdx

a= 65.75

b = 3.67

dx for 2007 = +7

2008 = 9

2009 = 11

= 65.75 + 3.67 (11) = 65.75 + 40.37 = 106.12

Examples 5 :

Find out straight line trend values from the following data (Population of punjab)

Year	-	1928	1938	1948	1958	1968	1978	1988	1998
Population (in Millions)	-	3.9	5.3	7.3	9.6	12.9	17.1	22.2	30.5

Solution : Computation of Trend values

x	**Y**	**dx**	**dx²**	**dxy**	**Yc**
1928	3.9	-7	49	-27.3	0.93
1938	5.3	-5	25	-26.5	4.55
1948	7.3	-3	9	-21.9	8.17
1958	9.6	-1	1	-9.6	11.79
		0	0	0	
1968	12.9	+1	1	12.9	15.41
1978	17.1	+3	9	51.3	19.03
1988	22.2	+5	25	111.0	22.65
1998	30.5	+7	49	213.5	26.27
N=8	108.8	0	168	303.4	

$$a = \frac{\in y}{N} = \frac{108.8}{8} = 13.6$$

$$b = \frac{\in dxy}{\in dx^2} = \frac{303.4}{168} = 1.81$$

Yc = a +b dx

1928 = 13.6 + 1.81 (-7)

= 13.6 - 12.67 = 0.93

1938 = 13.6 + 1.81 (-5)

= 13.6 - 9.05 = 4.55

1948 = 13.6 + 1.81 (-3)

= 13.6 - 5.43 = 8.17

1958 = 13.6 + 1.81 (-1)

= 13.6 - 1.81 = 11.79

1968 = 13.6 + 1.81 (1)

= 13.6 + 1.81 = 15.41

1978 = 13.6 + 1.81 (3)

= 13.6 + 5.43 = 19.03

1988 = 13.6 + 1.81 (5)

= 13.6 + 9.05 = 22.65

1998 = 13.6 + 1.81 (7)

= 13.6 + 12.67 = 26.27

Estimate for 2003 :

Yc = a +b dx

a= 13.6

b = 1.81

dx for 1998 = 7

2008 = 9

2003 = 8

2003 = 13.6 + 1.81 (8)

= 13.6 + 14.48

= 28.08

Examples 6 :

Find out straight line trend under least squares method.

Year	-	1989	1991	1992	1993	1994	1995	1998
Sales	-	140	144	160	152	168	176	180

Estimate the sales for 1996

Solution : Computation of Trend values

x	Y	dx	dx²	dxy	Yc
1989	140	-4	16	-560	139.45
1991	144	-2	4	-288	149.37
1992	160	-1	1	-160	154.33
1993	152	0	0	0	159.29
1994	168	+1	1	+168	164.25
1995	176	+2	4	+352	169.21
1998	180	+5	25	+900	184.09
13952	1120	1	51	412	

Note : In the above case assumed mean (starting point) is to be taken as under since there are breaks in the years. (The years are not continnous)

$$\frac{\text{Total of years}}{\text{No. of years}} = \frac{13952}{7} = 1993$$

Assumed maen shall be taken as 1993 (Starting point)

Trend Line equation Yc = a + b dx

Here, Since dx $\neq$ 0 (not equal to Zero), the values of Two constants a and b are to be found out by solving simultaneously the following two normal equations.

$\in y = Na + b. \in dx$ ....(1)

$\in dxy = \in dx.a + b \in dx^2$............................(2)

Substituting the respective values in the two equations , we get

1120 = 7a + b. 1 ....(1)

412 = 1a + 51 b ....(2)

7a + 1b = 1120 ....(1)

1a + 51b = 412 ....(2)

$7a + b = 1120$

$\underline{7a + 357b = 2884}$

$356b = 1764$

$$b = \frac{1764}{356}$$

$b = 4.96$

Substituting the above value of 'b' in the equation (1) we, get.

$7a + 4.96 = 1120$

$7a = 1120 - 4.96$

$7a = 1115.04$

$$a = \frac{1115.04}{7}$$

$a = 159.29$

Then the values of a and b must be substituted in the original formula (Yc = a +b dx) to find out the trend values.

Yc = a +b dx

1989 = 159.29 + 4.96 (-4)

= 159.29 - 19.84 = 139.45

1991 = 159.29 + 4.96 (-2)

= 159.29 - 9.92 = 149.37

1992 = 159.29 + 4.96 (-1)

= 159.29 - 4.96 = 154.33

1993 = 159.29 + 4.96 (0) = 159.29

1994 = 159.29 + 4.96 (1)

= 159.29 + 4.96 = 164.25

1995 = 159.29 + 4.96 (2)

= 159.29 + 9.92 = 169.21

1998 = 159.29 + 4.96 (5)

= 159.29 +24.8 = 184.09

Estimate for 1996 :

Yc = a +b dx

a= 159.29

b = 4.96

dx for 1995 = 2

1996 = 3

1996 = 159.29 + 4.96 (3)

= 159.29 + 14.88

= 174.17

Examples 7 :

The information relates to the annual sales in a retail shop.

Year -	2001	02	03	04	05	06	07
Sales (Rsin'000) -	120	130	135	125	145	150	140

From the above data find out -

1) straight line trend value under least squares method.

2) Monthly increase in sales.

3) Eliminate the trend values under Additive process of deseasonalisation and Multiplicative process of deseasonalisation method.

Solution : Computation of Trend values

x	**Y**	**dx**	**dx^2**	**dxy**	**Yc**
2001	120	-3	9	-360	123.21
2002	130	-2	4	-260	127.14
2003	135	-1	1	-135	131.07
2004	125	-0	0	0	135.00
2005	145	1	1	+145	138.93
2006	150	2	4	+300	142.86
2007	140	3	9	+420	146.79
N = 7	945	0	28	+110	

$a = \frac{\in y}{N} = \frac{945}{7} = 135$

$b = \frac{\in dxy}{\in dx^2} = \frac{110}{28} = 3.93$

Yc = a +b dx

2001 = 135 + 3.93 (-3)

= 135 -11.79 = 123.21

2002 = 135 + 3.93 (-2)

= 135 -7.86 = 127.14

2003 = 135 + 3.93 (-1)

= 135 - 3.93 = 131.07

2004 = 135 + 3.93 (0) = 135

2005 = 135 + 3.93 (1)

= 135 + 3.93 = 138.93

2006 = 135 + 3.93 (2)

= 135 +7.86 = 142.86

2007 = 135 + 3.93 (3)

= 135 +11.79 =146.79

Annual increase = b = 3.93

= 39300 Rs.

Monthly increase = $\frac{39300}{12}$

Rs = 327.5

Elimination of Trend values :

Additive process of deseasonalisation : $\in$ (Y - Yc) = 0

x	Y	Yc	Y -Yc
2001	120	123.21	-3.21
2002	130	127.14	2.86
2003	135	131.07	3.93
2004	125	135.00	-10.00
2005	145	138.93	6.07
2006	150	142.86	7.14
2007	140	146.79	-6.79
		$\in$ (Y - Yc) =	0

Multiplicative process of deseasonalisation : $\in$ (Y/ Yc) = N

x	Y	Yc	Y/Yc	
2001	120	123.21	120/123.21	= 0.97
2002	130	127.14	130/127.14	=1.03
2003	135	131.07	135/131.07	=1.03
2004	125	135.00	125/135.00	=0.93
2005	145	138.93	145/138.93	=1.04
2006	150	142.86	150/142.86	=1.05
2007	140	146.79	140/146.79	=0.95
N = 7			$\in$ (Y /Yc) = N=	7.00

5.3 MERITS OF LEAST SQUARES METHOD :

1) This method is completely free from personal bias of the analyst as it is very objective in nature. Any body using this method is bound to fit the same type of straight line and find the same trend values for the series.

2) Unlike the moving average method, under this method we are able to find the trend values for the entire time series without any exception for the extreme periods of the series even.

3) Unlike the moving average method, under this method it is quite possible to forecast any past or future values perfectly, since the method provides us with a functional relationship between two variables in the form of a trend line equation. Viz. Yc = a + b dx.

4) This method provides us with a rate of growth per period i.e. b with this rate of growth we can well determine the value for any past or previous year by the process of successive addition or deduction from the trend value of the origin of x.

5) This method provides us with the line of best fit form which the sum of the positive or negative deviation is zero and the sum of the squares of the deviations is the least i.e

   i) $\in$ (Y - Yc) = 0 and

   ii) $\in$ $(Y - Yc)^2$ = the least value

6) This method is the most popular and widely used for fitting mathematical function to a given set of observations.

7) This method is very flexible in the sense that it allows for shifting the trend origin from one point of time to another and for the conversion of the annual trend equation into monthly or quarterly trend equation and vice versa.

5.4 DEMERITS :

1) This method is very much rigid in the sense that if any item is added to or subtracted from the series it will need a thorough revision of the trend equation to fit a trend line and find the trend values there by.

2) In comparision to the other methods of trend determination, this method is a bit complicated in as much as it involves many mathematical tabulations, computations and solutions like those of simultaneous equations.

3) Under this method, we forecast the past and future values basing upon the trend values only and we donot take note of a seasonal, cyclical and irregular components of the series for the purpose.

4) This method is not suitable to business and economic data which conform to the growth curves.

5) It needs great care for the determination of the type of the trend curve to be fitted in viz. linear, parabolic, exponential or any other more complicated curve. An erratic selection of the type of curve may lead to fallacious conclusions.

6) This method is quite appropriate for both very short and very long series. It is also unsuitable for a series in which the difference between the successive observations are not found to be constant or nearly so.

5.5 SUMMARY :

Computation of trend values under least squares method is a scientific technical and correct method even though it is some what difficult to calculate.

5.6 QUESTIONS :

1. Explain the method of computation of trend values under least squares method.
2. What is the significance of Least Squares Method ?
3. Why this method is called Least Squares Method ?
4. What are the merits of Least Squares Method ?
5. What are the Demerits of Least Squares Method ?

5.7 EXERCISES :

1. Find out the trend values under least squares method from the following data.

Year -	2003	04	05	06	07
Sales (Rs in cores) -	12	18	20	23	27

Estimate the sales for the year 2009

2. Fit a straight line trend equation under least squares method from the following data.

Year -	2001	02	03	04	05	06	07
Profits (Rs.in cores) -	120	144	150	130	160	170	190

Plot the data and trend values on a graph paper.

Estimate the profits for the year 2010

3. Find out the trend values under least squares method from the data given below.

Year -	2003	04	05	06	07
Number -	38	38	46	40	56

Estimate the values for the year 2011 and 1999.

4. Find out straight line trend values under least squares method. Estimate the values for the year 2010.

Year -	2001	02	03	04	05	06	07
Values -	100	120	110	140	80	95	115

5. From the following data find out the trend values under least squares method. Show the data on a graph paper

Year -	2002	03	04	05	06	07
Values -	27	30	35	40	44	45

6. Fit a straight line trend equation under least squares method from the following data.

Year -	2001	02	03	04	05	06	07
Production (in M.T) -	25	27	32	36	44	55	69

Estimate the sales for the year 2008

7. Find out straight line trend values under least squares method and estimate the production for the year 1973

Year -	1966	68	69	70	71	72	75
Production (in M.T) -	77	88	94	85	91	98	90

8. Find out the trend values under least squares method

Year -	1990	92	95	96	97	99	2000
Price (in Rs.) -	25	32	40	37	44	50	57

Estimate the price for the year 2003

9. From the following data find out the trend values under least squares method and estimate the population for the year 1992.

Year -	1980	81	84	85	87	88	89
Population -	2.5	2.9	3.2	3.0	4.7	5.4	6.3

10. Find the trend values under least squares method

Year -	2001	02	03	04	05	06	07
Production (Rs.000) -	360	390	405	375	435	450	420

Find out the monthly increase in production

Eliminate the trend values under Additive and Multiplicative methods.

11. From the following data, find out the trend values under least squares method. Eliminate the trend values under Additive and Multiplicative process.

Year	-	1999	2000	01	02	03	04	05	06	07
Values	-	380	400	650	720	690	620	670	950	1040

12. Find out the trend values under least squares method. Prove that $\in (Y - Yc) = 0$

Year	-	2004	2005	2006	2007
Sales (in '000)	-	10	13	15	12

13. Find out the trend values under least squares method.

Year	-	2001	02	03	04	05	06	07
Production (000 units)	-	100	105	109	96	102	108	112

show the data and trend values on a graph paper

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Lesson - 6

INDEX NUMBERS

6.0 Objective :

After studying this lesson you should be able to understand

1. Meaning of Index numbers
2. Features, Advantages of Index Numbers
3. Types of Index numbers

Structure of Lesson :

6.1 Meaning of Index Numbers

The index number is a specialised average designed to measure the change in a group of related variables over a period of time. It gives a central idea of changes in a particular phenomenon viz price, quality, value etc., over a period of time. Historically, the first index was constructed in 1764 to compare the Italian price index in 1750. Today index numbers have become one of the most widely used statistical devices and there is hardly any field where they are not used. Almost all the fields of human activities viz., business, economics, industries and Commerce use index numbers to find out the gist of the changes in any phenomenon over a period of time. Because of its importance, economists describe index numbers as a *barometer of economic activities*. However because of its significance, it has been defined *variously* by various authors. Some of the definitions are quoted here.

6.2 Definitions

1. According to 'croxton & cowden' "Index numbers are devices for measuring differences in the magnitude of a group of related variables".

2. According to Horace Sacrist, "Index numbers are a series of numbers by which changes in the magnitude of a phenomenon are measured from time to time or place to place".

3. According to 'Morris Hamburg' "In its simplest form, an index number is nothing more than a relative number, or a relative which expresses the relationship between two figures, where one of the figures is used as a base".

4. According to G.Simpson and F.Kafka, "Index numbers are today one of the most widely used statistical devices. They are used to feel the pulse of the economy and they have come to be used as the indicators of deflationary tendencies".

6.3 Characteristics or Features of Index Numbers

From the above definitions, the salient characteristics of index numbers can be brought out as under.

1. Specialized Averages : Index numbers are used for purposes of comparison in situations where two or more series are expressed in different units or the series are composed of different types of items.

For example : While constructing a consumer price index the various items are divided into broad heads, namely food, clothing, fuel and lighting, house rent and miscellaneous. These items are expressed in different units (kgs, meters, etc.) An average of all these items expressed in different units is obtained by using the technique of index numbers.

2. Measures the net change : Index numbers measure the net change in a group of related variables. Since index numbers are essentially averages, they describe in one single figure the increase or decrease in a group of related variables under study.

For example : If the consumer price index of working class for Delhi has gone up to 113 in 1986 compared to 1985. It means that there is a net increase of 13% in the prices of commodity included in the index.

3. Measures the effect of changes over a period of time : Index numbers are most widely used for measuring changes over a period of time and compare economic conditions of different locations, different industries, different cities or different countries.

For example : We can compare the agricultural production, industrial production, imports, exports, wages, etc., at two different times.

6.4 Uses or Advantages of Index numbers

Index numbers are indispensable tools of economic and business analysis. They are described as 'barometers of economic activity. In modern days index numbers are extensively used for a large number of purposes in a variety of fields viz. business, industry, economics and politics etc., Their significance can be best appreciated by the following points.

1. Useful in framing suitable policies : Many of the economic and business policies are guided by index numbers.

For Example: to decide the increase in dearness allowance of the employees, the employers depend upon the cost of living index.

2. Useful in Comparison : Index numbers useful to compare a person's intelligence score with that of an average for his or her age. Health authorities prepare indices to display change in the adequacy of hospital facilities and the educational institutes have devised formula to measure changes in the effectiveness of school systems.

3. Useful in revealing trends and tendencies : Since index numbers are most widely used for measuring changes over a period of time the time series so formed enable us to study general trend of the phenomenon under study.

For Example: Index number of imports for the last 8 years may show an upward trend or downward trend.

4. Useful in deflating : Index numbers are used to adjust the original data for price changes or to adjust wages for cost of living changes and thus transform nominal wages into real wages.

5 Useful to measure wholesale price level : Index numbers are used to measure the changes in the wholesale price level of a country over a period of time.

6. Useful in the fixation of wages : Index numbers are useful to adjust salaries of employees according to the cost of living.

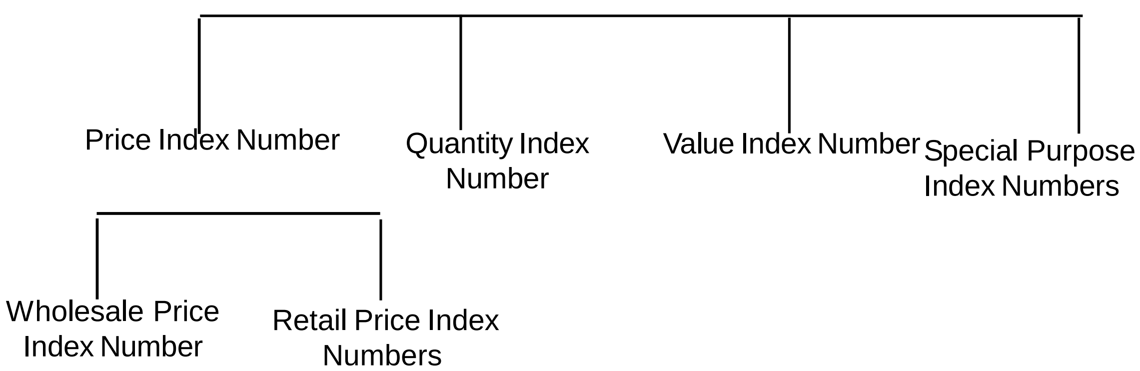
7. Useful in making decisions : The index numbers provide some guideposts that one can use in making decisions.

6.5 Classification of Index Numbers

Index numbers may be classified in terms of what they measure. In economics and business the classification is

1. Price Index numbers
2. Quantity Index numbers
3. Value Index numbers
4. Special purpose Index numbers

Classification of Index Numbers



6.5.1 Price Index Numbers : Price Index numbers measure and permit comparison of the price of certain goods. These are most widely used in economic and business sectors. Price Index is the total of current year prices for th various commodities in question is divided by the total of base year prices and the quotient is multiplied by 100. Price Index numbers can be grouped under two heads.

a. Wholesale Price Index Numbers

b. Retail Price Index Numbers

a. Wholesale Price Index Numbers

Wholesale price index numbers are used to study the price levels.

b. Retail Pirce Index Numbers

Retail price index numbers are used to measure the effect of changes in the price of a set of goods and serivces on the purchasing power of a particular community of people during a given period. This is also otherwise called *Consumer Price Index Number.* Such index numbers are constructed separately for different classes of people viz., govt., employees, factory workers, agricultural labour, etc., to speak of the effect of rise and fall in the prices of the commodities on their cost of living.

6.5.2 Quantity Index Numbers : Quantity or volume index numbers is one that measures the changes in the level of quantities consumed during a given period with reference to a base period. Just like the price indices the quantity indices can be constructed under the various methods. In such index number, the prices are treated as weight and the quantities are treated as the variables. In constructing quantity index numbers, the problems confronting the statistician are analogous to those involved in price index.

6.5.3 Value Index Numbers : Value index numbers are used to measure the changes in values of products. In such index number total of the values of the commodities consumed, produced or marketed during the current yer are divided by total of the values of the commodities consumed, produced or marketed during the base year. It is expressed as percentage by multiplying with 100.

6.5.4 Special Purpose Index Numbers : Some index numbers are constructed for special purposes. Such index numbers are called special purpose index numbers. For instance the different employers including the governments use these index numbers as an instruments in fixing the dearness allowances and bonus etc. These ar ealso known as cost of living index numbers.

6.6 Exercise

6.6.1 Short Questions

For the following questions write answers in brief.

1. What is an Index numbers?

2. Define Index numbers.

3. What is Price Index?

4. What is Quantity Index?

5. What is Value Index?

6.6.2 Essay Questions

1. Explain the importance of Index numbers in Business and Economic sectors.

Or

'Index numbers are Economic Barometers' – Discuss.

2. Explain features of Index numbers.

Dr. K. Kanaka Durga

LESSON 7:

CONSTRUCTION OF INDEX NUMBERS

7.0 OBJEC\TIVE

After studying this lesson you should be able to understand.

1. Problems in the construction of Index Numbers.

2. Different methods of Construction of Index Numbers.

STRUTURE OF LESSON

7.1. INTRODUCTION

The term Index Numbers literally means a numerical figure that indicates the value of a variable effected at a particular time in terms of percentage of its value that stood on a different time. It is a special type of average that gives a central idea of changes in a particulr phenomenon viz. Price, quantity value, etc., over a period of time. Some economists describe it as a barometer of price level changes or any other economic activity. The idea of computing the Index numbers came into being for the first time in 1764 in Italy. Now, it is being used as a formidable instrument by the statisticians in almost all the fields of human activities viz., business, economics, industries and commerce to findout the gist of the changes in any phenomenon over a period of time.

7.2 PROBLEMS IN THE CONSTRUCTION OF INDEX NUMBERS

The Construction of an index number involves a lot of problems for a staistician. Some of the importan problems may be listed as under :

1. Purpose of the index number
2. Selection of the base year
3. Selection of the items
4. Selection of the data about price quotations
5. Selection of appropriate Weights
6. Selection of suitable method of averaging
7. Choice of the appropriate formula

7.2.1 THE PURPOSE OF THE INDEX NUMBER

The purpose of constructing the Index must be very clearly dicided

1. What is the index to measure

3. Why it is to be measured

There are many purposes for which an index number may be constructed. For instance, the purpose may be to measure the general price level changes in a country. The set of data relating to items, price quotations etc., to be included in the constuction of index number. The data should be collected according to the pupose of the index number.

7.2.2 SELECTION OF BASE YEAR

Whenever Index numbers are constructed a reference is made to some base period. The base period of an index number is the period against which comparisions are made. It may be a year, month or a day. The index for base period is always taken as 100. The following points need careful consideration while selecting a base yaer.

A. The base peiod should be a normal one:

The period that is selected as base should be free from abnormalities like wars, earth-quakes, famines etc., when the statistician is not able to find any such year, in that case he should take average of the values of the relevant data relating to a number of years as the value of the base year.

B. The base period should not be too distant in the past :

It is desirable to select a recent year as a base period which is more helpful for the comparisons

C. Fixed base or Chain base :

In the fixed base method, the year or the period of years to which all other prices are related is constant for all times.

In the chain base method the prices of a year are linked with those of preceding year.

7.2.3 SELECTION OF THE NUMBER OF ITEMS

The next problem which a statistician faces in the construction of an index number is the selection of the items and their varieties. This problem should be carefully tackled keeping in view the objecive for which the index number is being constructed. However, the number of items and their varieties to be included in an index number, should neither be too large nor too small. The commodities should be selected in such a manner that they are representatives of the tastes, habits and customs of the people for whom the index is meant.

7.2.4 SELECTION OF THE DATA ABOUT PRICE QUOTATIONS

After the commodities have been selected, the next problem is to obtain price quotations for these commodities, prices of many commodities vary from place to place and even from shop to shop in the same market. It is not practicable to obtain price quotations from all the shops. A selection must be made of representative places and persons. These places should be those which are well known for trading of those commodities. Price quotations can be obtained through appointed persons or agencies. There are two methods of quoting prices -

1. Money Prices Ex. Sugar Rs.20 per kg.

2. Quantity Prices. Ex. 1/2 kg for Rs. 10.

A decision must also be made as to whether the wholesale prices or retail prices are required. The choice would depend upon the purpose of the index.

7.2.5 SELECTION OF APPROPRIATE WEIGHTS

The next problem which a statistician faces in the construction of an index number is the assignment of weights. The term weight refers to the relative importance of the different itmes in the construction of the index. hence it is necessary to devise some suitable method to allocate weights. There are broadly two types of indices -

1. Unweighted Indices (Weights are not specific)

2. Weighted Indices (Weights are specific)

Weights are two types -

1. Quantity Weights:

It is symbolised by q, means the amount of commodity produced, distributed, or consumed in some time period.

2. Value Weight:

It is symbolised by the combination of price and quantity produced, distributed or consumed,

i.e., p x q. (p = price, q = quanity)

7.2.6 SELECTION OF SUITABLE METHOD OF AVERAGING

Since index numbers are specialized averages a decision has to be made as to which particular average i.e., arithemetic mean, median, mode, geometric mean or harmonic mean should be used for constructing the index. Geometric mean is the best average in the construction of index numbes, because it gives equal weights to equal ratio of change. But arithmetic mean is more popularly used while constructin index numbers because it is much more simple to compute than the geometric mean.

7.2.7 CHOICE OF THE APPROPRIATE FORMULA

A large number of formulae have been devised for consructing the index. The selection of the appropriate formula would depend not only on the purpose of the index but also on the data available. Prof. Irving Fisher has suggested that an appropriate index is that which satisfied time reversal test and factor reversal test.

7.3. METHOD OF CONSTRUCTING INDEX NUMBERS

A large number of formula have been devised for constructing index numbers. Broadly speaking, they can be grouped under two heads:

a. Unweighted indices

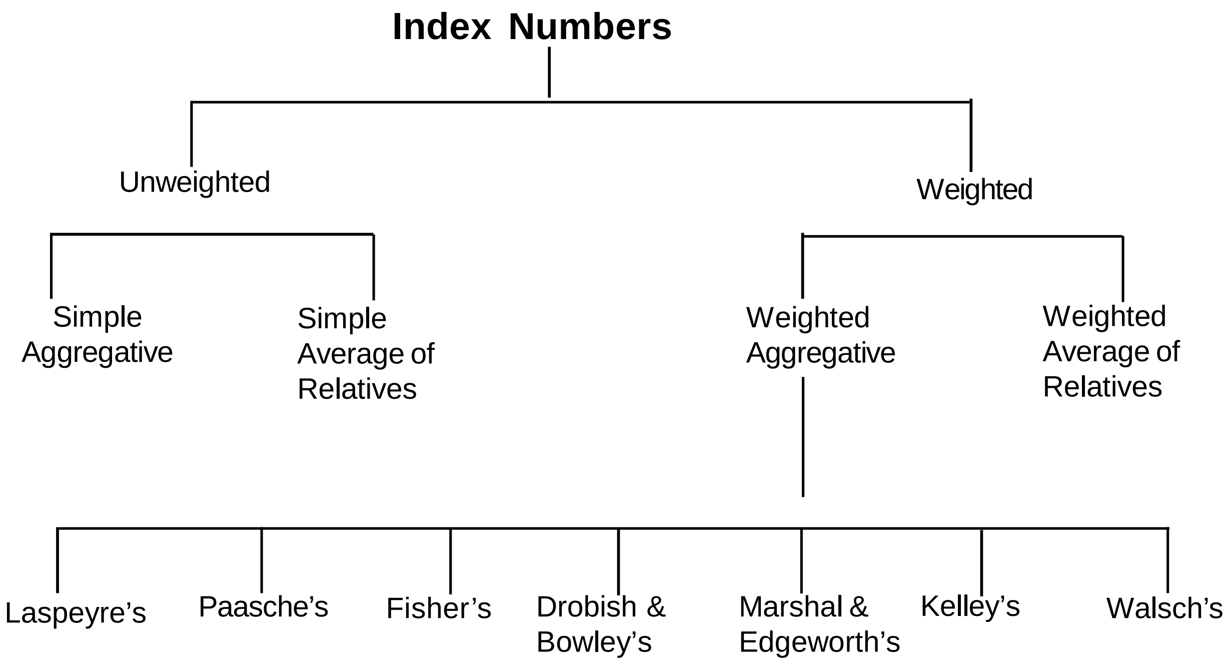
b. Weighted indices.

In he unweighted indices weights are ot expressly assigned whereas in the weighted indices weights are assinged to the various items. Each of these types may be further divided under two heads:

1. Simple Aggregative

2. Simple Average of Relatives

The following chart illustrates the various methods.



7.4. UNWEIGHTED INDEX NUMBERS

7.4.1 Simple Aggregative Method

This is the simlext method of consructing index numbers. when this method is used to consruct a price index the total of current year prices for the various commodities in questio is divided by the total of base year prices and the quotient is multiplied by 100.

The formula is

$$Po_1 = \frac{\Sigma P_1}{\Sigma P_0} x100$$

Po_1 = Price Index of the current year with reference to the base year

ΣP_1 = Total of the prices of the current year.

ΣPo = Total of the prices of the base year.

The merits and demerits of this method can be outlined as under.

Merits

1. It is simple to understand
2. It is easy to calculate

Demerits

1. It is based on he assumption that the items and their prices are expressed in he same unit.
2. It ignores the relaive importance of the different commodities included in the index number.

3. It is not capable of being calculated through geometric mean and median.

4. It is affected by the magnitude of the prices of the different commodities.

ILLUSTRATIONS

Illustration - 1

From the following data construct an index for 2006 taking 2005 as base.

Commodity	Price in 2005	Price in 2006
	Rs.	Rs.
M	100	140
N	80	120
O	160	180
P	220	240
Q	40	40

Solution:

Construction of Price Index

Commodity	Price in 2005	Price in 2006
	P_O	P_1
M	100	140
N	80	120
O	160	180
P	220	240
Q	40	40
	600	720

$$Po_1 = \frac{\Sigma P_1}{\Sigma P_0} x100$$

ΣP_O = 600 - Total of prices of 2005

ΣP_1 = 720 - Total of the prices of 2006

$$PO_1 = \frac{\Sigma P_1}{\Sigma P_0} x100$$

$$PO_1 = \frac{720}{600} \times 100 = 120$$

It reveals that there is an increase of 20% in the prices of 2006 comapre to 2005.

Illustration - 2

From the following data computer the price index for 2006 on the basis of 2003 prices..

Items	Unit	Price in 2003 Rs.	Price in 2006 Rs.
Rice	Quintals	500	600
Dal	Kg	15	20
Vegetables	Kg	6	8
Meat	Kg	40	50
Fish	Kg	30	40
Milk	Litre	4	7
Clothing	Metre	25	30

Solution:

Construction of Price Index

Items	Unit	Price in 2003 Rs.	Price in 2006 Rs.
Rice	Quintals	500	600
Dal	Kg	15	20
Vegetables	Kg	6	8
Meat	Kg	40	50
Fish	Kg	30	40
Milk	Litre	4	7
Clothing	Metre	25	30
		620	755

$$Po_1 = \frac{\Sigma P_1}{\Sigma P_0} x 100$$

$\Sigma P_O = 755$

$\Sigma P_1 = 620$

$$PO_1 = \frac{755}{620} \times 100 = 121.77$$

Illustration - 3

From the following data compute the price index for 2003 on the basis of 2006 prices..

Item	Unit	Price in 2003 Rs.	Price in 2006 Rs.
A	Kg	5	6
B	Kg	15	20
C	Litre	6	8
D	Litre	40	50
E	Litre	30	40
F	Kg	4	7
G	Metre	25	30

Solution:

Construction of Price Index

Items	Unit	Price in 2003 Rs.	Price in 2006 Rs.
A	Kg	5	6
B	Kg	15	20
C	Kg	6	8
D	Litre	40	50
E	Litre	30	40
F	Kg	4	7
G	Metre	25	30
		125	161

$$Po_1 = \frac{\Sigma P_1}{\Sigma P_0} x100$$

$\Sigma P_O = 125$

$\Sigma P_1 = 161$

$$PO_1 = \frac{161}{125} \times 100 = 128.8$$

$PO_1 = 128.8$

Illustration - 4

From the following data relating to prices of different items, calculate the price indices for 2005 and 2006 on the basis of the prices of 2004.

Item	Unit	Prices in 2004	Prices in 2005	Prices in 2006
		Rs.	Rs.	Rs.
Food	Kg	20	25	30
Fuel	Kg	40	45	50
Rent	P.M.	3000	3500	4000
Clothes	Metre	100	120	130
Fruits	Dozen	60	65	70

Solution:

Construction of Price Index

Item	Unit	Prices in 2004	Prices in 2005	Prices in 2006
		Rs.	Rs.	Rs.
Food	Kg	20	25	30
Fuel	Kg	40	45	50
Rent	P.M.	3000	3500	4000
Clothes	Metre	100	120	130
Fruits	Dozen	60	65	70
		3220	3755	4280

$$Po_1 = \frac{\Sigma P_1}{\Sigma P_0} x100$$

For 2005 = $P0_1 = \frac{3755}{3220} = 1.66$

For 2006 = $P0_1 = \frac{4280}{3755} = 1.13$

Illustration - 5

For the data given below, calculate the index number by taking :

1. 1998 as the base year

2. 2005 as the base year

3. Shifting 1998 to 2000 as the base year

Year	Price of Commodity - 'X'
1998	4
1999	5
2000	6
2001	7
2002	8
2003	10
2004	9
2005	10
2006	11

Solution : 1. Index numbers taking 1998 as the base year.

Year	Price of Commodity - 'X'	Index numbers 1998 = 100
1998	4	4/4 X 100 = 100
1999	5	5/4 X 100 = 125
2000	6	6/4 X 100 = 150
2001	7	7/4 X 100 = 175
2002	8	8/4 X 100 = 200
2003	10	10/4 X 100 = 250
2004	9	9/4 X 100 = 225
2005	10	10/4 X 100 = 250
2006	11	11/4 X 100 = 275

2. Index numbers taking 2005 as the base year.

Year	Price of Commodity - 'X'	Index numbers(2005 = 100)
1998	4	4/10 X 100 = 40
1999	5	5/10 X 100 = 50

2000	6	6/10 X 100 = 60
2001	7	7/10 X 100 = 70
2002	8	8/10 X 100 = 80
2003	10	10/10 X 100=100
2004	9	9/10 X 100 = 90
2005	10	10/10 X 100=100
2006	11	11/10 X 100=110

3. Index numbers shifting 1998 to 2000 as the base year period.

Average Base Year = $\frac{4+5+6}{3} = \frac{15}{3} = 5$

Hence 1999 will be taken as 100.

Year	Price of Commodity - 'X'	Index numbers(1999 = 100)
1998	4	4/5 X 100 = 80
1999	5	5/5 X 100=100
2000	6	6/5 X 100 = 120
2001	7	7/5 X 100 = 140
2002	8	8/5 X 100 = 160
2003	10	10/5 X 100=200
2004	9	9/5 X 100 = 180
2005	10	10/5 X 100=200
2006	11	11/5 X 100=220

7.4.2 SIMPLE AVERAGE OF PRICE RELATIVES METHOD

When this method is used to construct a price index, first of all price relatives are obtained for the various items included in the index and then average of these relatives is obtained using any one of the measures of central value, i.e., arithmetic mean, median, mode, geometric mean or harmonic mean.

7.4.2.1 Index Number through Arithmetic Mean

When Arithmetic Mean is used to construct Index number, the principle used is

$$Po_1 = \frac{\Sigma I}{N}$$

Where, Po_1 = Prices Index of the Current Year

I = Prices relative of the respective items

N = Total number of items

$I = \frac{P_1}{P_0} \times 100$

P_1 = Price of the current year

P_0 = Price of the base year

7.4.2.2 Index Number through Geometric Mean

$$P0_1 = A.L. \frac{\Sigma \log I}{N}$$

A. L = Anti log.

Σ log I = Total of logs of Price relatives

N = Total Number of Items

THE MERITS AND DEMERITS OF THIS METHOD CAN BE ENUMERATED UNDER :

Merits

1. It is simpel to understand and easy to calculate.
2. It gives equal weights to all the items.
3. It is not effected by the magnitude of the prices of the various items.
4. It satisfies the unit test in the sense that it is not affected by the units of price quotations.

Demerits

1. Difficulty is faced with regard to the selection of an appropriate average.
2. The use of geometric mean involves difficulties of computation.
3. It assumes that all the relatives are of equal importance which is highly objectionable from economic point of view.

Illustration - 6

From the following data construct an index for 2006 taking 2005 as base by the average of relatives method, using :

a. Arithmetic Mean

b. Geometric Mean

a) **Using Arithmetic Mean**

Commodity	Prices in 2005(Rs.) P_0	Prices in 2006(Rs.) P_1	Price Relatives $P_1 / P_0 \times 100$
P	50	70	140.0
Q	40	60	150.0
R	80	90	112.5
S	110	120	109.1
T	20	20	100.0
			611.6

$$P0_1 = \frac{\Sigma I}{N}$$

$$\Sigma I = \frac{P_1}{P_0} x100$$

$\Sigma I =$ 611.6

N = 5

$$P0_1 = \frac{611.6}{5} = 122.32$$

$P0_1 = 122.32$

b) **Using Geometric Mean**

Commodity	Prices in 2005(Rs.) P_0	Prices in 2006(Rs.) P_1	Price Relatives $P_1 / P_0 \times 100$	Log P
P	50	70	140.0	2.1461
Q	40	60	150.0	2.1761
R	80	90	112.5	2.0512
S	110	120	109.1	2.0378
T	20	20	100.0	2.0000
				10.4112

$$P0_1 = \text{A.L.} \frac{\Sigma \log P}{N}$$

$\Sigma \log P = 10.4112$

N = 5

$P0_1$ = A.L. $\frac{10.4112}{5}$

$P0_1$ = A.L. of 2.0822 = 120.9

$P0_1$ = 120.9

c) **Using Median**

Some economists, notably .Y.Edgeworth have preferred to use the median which is not affected by extreme values.

Illustration-7 :

From the followign data construct the index number by simple relative method for 2006 taking 2003 as the base year using :

i) Arithmetic Mean

ii) Geometric Mean

iii) Median

Commodites	P	Q	R	S	T
Prices in 2003	20	30	10	25	40
Prices in 2006	25	30	15	35	45

Solution :

Construction of Index Number for 2006 (Base year 2003, Using Arithmetic Mean)

Commodites	Prices in 2003(P_0)	Prices in 2006(P_1)	Price Relatives(I)
P	20	25	25/ 20 x 100 = 125
Q	30	30	30/ 30 x 100 = 100
R	10	15	15/ 10 x 100 = 150
S	25	35	35/ 25 x 100 = 140
T	40	45	45/ 40 x 100 = 125
			640

$$P0_1 = \frac{\Sigma I}{N}$$

$\Sigma I = \frac{P_1}{P_0} x100$

$\Sigma I =$ 640

N = 5

$P0_1 = \frac{640}{5} = 128$

$P0_1 = 128$

d) **Construction of Index Number for 2006 using Geometric Mean**

Commodites	Prices in 2003(P_0)	Prices in 2006(P_1)	Price Relatives(I)	Log I
P	20	25	125	2.0969
Q	30	30	100	2.0000
R	10	15	150	2.1761
S	25	35	140	2.1461
T	40	45	125	2.0969

10.5160

$P0_1 = \text{Anti Log of } . \frac{\Sigma \log I}{N}$

Σ log I = 10.5160

N = 5

$P0_1 = \text{A.Log } \frac{10.5160}{5} = 2.1032$

$P0_1$ = Anti Log of 2.1032 = 126.82

$P0_1$ = 126.82

e) **Construction of Index Number for 2006 (Using Median)**

Commodites	Prices in 2003(P_0)	Prices in 2006(P_1)	Price Relatives(I)	Price relating in ascending order
P	20	25	125	100
Q	30	30	100	125
R	10	15	150	125
S	25	35	140	140
T	40	45	125	150

$P0_1$ = Value of $\frac{(N + 1)}{2}$ th item

N = 5

Value of $\frac{(5 + 1)}{2}$ th item

Value of $\frac{6}{2}$ th item = 3rd item

3rd item = 125

$P0_1$ = 125

Illustration-8

Fom the following data construt an Index for 2006 taking 2005 as base by the average of price relative method using (a) Arithmetic Mean (b) Geometric Mean

Commodity	A	B	C	D	E	F
Prices in 2005	40	60	20	50	80	110
Prices in 2006	50	60	30	70	90	110

Solution :

Commodity	Prices in 2005 (Rs.) P_0	Prices in 2006 (Rs.) P_1	Price Relatives $I = P_1/P_0 \times 100$	Log I
A	40	50	125	2.0969
B	60	60	100	2.0000
C	20	30	150	2.1761
D	50	70	140	2.1461
E	80	90	112.5	2.0511
F	110	110	100	2.0000
			727.5	12.4702

a) **Arthimetic Mean** :

$$P0_1 = \frac{\Sigma I}{N}$$

$$\Sigma I = \frac{P_1}{P_0} x100$$

$\sum I = 727.5$

$N = 6$

$P0_1 = \frac{727.5}{6} = 121.25$

$P0_1 = 121.25$

b) **Geometric Mean** :

$$P0_1 = \text{Anti Log of } . \frac{\Sigma \log I}{N}$$

$\sum \log I = 12.4702$

$N = 6$

$P0_1 = \text{A.Log } \frac{12.4702}{6} = 2.0784$

$P0_1 = \text{Anti Log of } 2.0784 = 119.8$

$P0_1 = 119.8$

7.5 WEIGHTED INDEX NUMBERS

Construction of useful index numbers requires a conscious effort to assign to ech commodity a weight in accordance with its importance in the total phenomenon that the index is supposed to describe. Weighted Index numbers are two types :

Weighted Index Numbers
- Weighted Aggregative Indices
- Weighted Average of Relative Indices

7.5.1 WEIGHTED AGGREGATIVE INDEX NUMBERS

These indices are of the simple aggregative type wih the fundamental difference that weights are assigned to the various items included.in the Index. Following are the various methods to construct Index numbers to assigning weights.

Weighted Aggregative Index Numbers
- Laspeyre's Method
- Paasche Method
- Dorbish and Bowley Method
- Fisher's Method
- Marshall - Edgeworth Method
- Kelley's Method

Laspeyres Method :

The Laspeyres Price Index is a weighted aggregate price index, where the weights are determined by quantities in the base period. The formula for constructing the index is :

$$P0_1 = \frac{\sum p_1 q_0}{\sum p_0 q_0} x100$$

$P0_1$ = Index Number

$\sum P_1Q_0$ = Multiply the current year prices of various commodities with base year wieghts.

$\sum P_0Q_0$ = Multiply the base year prices of various commodities with base year weights.

P_1 = Base Year Price

P_0 = Current Year Price

Q_0 = Base Year Quantity

Q_1 = Current Year Quantity

Laspeyres Index attempts to get change in aggregate value of the base period list of goods when valued at given period prices.

Merits :

1. It is easy to understand and simle to calculate.
2. This index is very widely used in practical work.
3. It satisfies the unit test of adequacy.

Demerits :

1. It does not take into consideration the consumption pattern.
2. It has an upward bias in weighting the commodities. When with the rise in the prices people reduce their quantities of consumption the weights remain fixed being assigned on the basis of the quantities of the base year.
3. It does not permit the use of any other averages viz. Geometric mean, Median, etc.

Illustration -9 :

Construct Laspeyers Index Number from the following data.

Commodity Base Year Current Year

	Quantity	Price	Quantity	Price			
	Q0	P0	Q1	P1	P_1Q_0	P_0Q_0	
Bread	6		40	7	30	180	240
Meat	4		45	5	50	200	180
Tea	5		90	15	50	250	450
					630	870	

$$P0_1 = \frac{\sum p_1 q_0}{\sum p_0 q_0} x100$$

$\sum P_1Q_0 = 630$

$\sum P_0Q_0 = 870$

$$P0_1 = \frac{630}{870} \text{ x } 100 = 72.41$$

7.5.2 PAASCHE'S METHOD :

The Paasche price index is a weighted aggregate price index in which the weights are determined by quantities in the given year. The formula for constructing the index is

$$P0_1 = \frac{\sum p_1 q_1}{\sum p_0 q_1} x100$$

$\sum P_1Q_1$ = Sum of multiplications of current year prices of commodities with current year weights.

$\sum P_1Q_1$ = Sum of multiplications of base year prices of commodity with current year weights.

This formula answers the question, what would be the value of the given period list of gods when valued at base period prices ?

Merits :

1. This method is very easy to understand and simple to calculate.

2. It is based on weights which remain fixed for both the current and base year.

3. It satisfies the unit test of adequancy of a formula of index number.

Demerits :

1. It assumes the quantities consumed in the current year to be the quantities consumed in the base year. This assumption may not always hold good.

2. It does not satisfy the Time Reversal Test, Factor Reversal Test and Circular Test of adequancy.

Illustration-10 :

From the following data construct Paasche's Index Number :

Commodity	Base Year Quantity Q_0	Base Year Price P_0	Current Year Quantity Q_1	Current Year Price P_1
A	12	10	15	12
B	15	7	20	5
C	24	5	20	9
D	5	16	5	14

Solution :

Commodity	Base Year Quantity Q_0	Base Year Price P_0	Current Year Quantity Q_1	Current Year Price P_1	P_1Q_1	P_0Q_1
A	12	10	15	12	180	150
B	15	7	20	5	100	140
C	24	5	20	9	180	100
D	5	16	5	14	70	80
					530	470

$$P0_1 = \frac{\sum p_1 q_1}{\sum p_0 q_1} x100$$

$\sum P_1Q_1 = 530$

$\sum P_1Q_1 = 470$

$$P0_1 = \frac{530}{470} \times 100 = 112.75$$

$P0_1 = 112.75$

7.5.3 KELLEY'S METHOD

Mr.Kelley, in his method takes the quantity of any year or average of quantiies of any number of years as the weights of the items. For this reason, the method is also otherwise known as the method of fixed weight aggregative index. the formula is

$$P0_1 = \frac{\sum p_1 q_0}{\sum p_0 q_1} x100$$

q = $\frac{q_0. + q_1}{2}$ or $\sqrt{q_0.q_1}$

q = Average of two years Quantity

Illustration-11

Construction of Kelley's Index number

Commodity	Base Year		Current Year			
	Price	Quantity	Price	Quantity		
	P_0	Q_0	P_1	Q_1	P_1Q	P_0Q
P	10	49	12	50	29400	24500
Q	12	25	15	20	7500	6000
R	18	10	20	12	2400	2160
S	20	5	40	2	40	200
						39700

7.5.4 MARSHALL - EDGEWORTH METHOD

In this method also both the current year as well as base year prices and quantity are consideed. The formula for constructing the Index is :

$$po_1 = \frac{\sum p_1 q_0 + \sum p_1 q_1}{\sum p_0 q_0 + \sum p_0 q_1} x100$$

Illustration -12:

From the following data construct index number using Marshall - Edge worth method.

Commodity	Prices in 2005	Prices in 2006	Quantity in 2005	Quantity in 2006
A	200	180	10	12
B	20	23	22	20
C	25	30	18	14
D	5	7	30	24

Commodity	Prices in 2005	Prices in 2006	Quantity in 2005	Quantity in 2006	P_1q_0	P_1q_0	P_1q_0	P_1q_0
A	200	180	10	12	1800	2160	2000	2400
B	20	23	22	20	506	460	440	400
C	25	30	18	14	540	420	450	350
D	5	7	30	24	210	168	150	120
					3056	3208	3040	3270

$$po_1 = \frac{\sum p_1q_0}{\sum p_0q_0} + \frac{\sum p_1q_1}{\sum p_0q_1} x100$$

$\sum p_1q_0 = 3056$, $\sum p_1q_0 = 3208$, $\sum p_0q_0 = 3040$, $\sum p_0q_1 = 3270$

$$po_1 = \frac{3056}{3040} + \frac{3208}{3270} x100$$

$$po_1 = \frac{6264}{6310} x100 = 99.27099$$

7.5.5 DORBISH AND BOWLEY'S METHOD :

Dorbish and Bowley's have suggested simple Arithmetic mean of the two indices Laspeyres and Paasche. This Index number takes into account the influence of both the periods i.e., current as well as base periods. The formula for constructing the Index is

$$Po_1 = \frac{L + P}{2}$$

L = Laspeyres

P = Paasche

or

$$po_1 = \frac{\sum p_1q_0}{\sum p_0q_0} + \frac{\sum p_1q_1}{\sum p_0q_1} x100$$

7.5.6 FISHER'S IDEAL INDEX :

The method divised by Prof. Irving Fisher for construction of an Index number is known after his name as Fisher's aggregative method. Prof Fisher has given a number of formulae for constructing index numbers and of there he calls one as the ideal index.

The formula is

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$$

or

$$po_1 = \sqrt{LxP}$$

L = Laspeyres Index

P = Paasche Index

Fisher's Ideal Index is the geometric mean of the Laspeyres and Paasche Indices.

Merits :

1. Fisher's Index is known as 'Ideal' because, it is based on the geometric mean which is theoretically considered to be the best average for constructing index numbers.

2. It takes ino account both current year as well as base year prices and quantities.

3. It satisfies both the time reversal test as well as the Factor Reversal Test.

Demerits :

1. It is excessively laborious.

2. It is not simple to understand by a common man.

Illustration -13: Construct Fisher's Ideal Index Number from the following data :

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	8	50	20	60
B	2	15	6	10
C	1	20	2	25
D	2	10	5	8
E	1	40	5	30

Solution :

Commodity	Base Year		Current Year		P_1q_1	P_0q_0	P_1q_0	P_0q_1
	Price	Quantity	Price	Quantity				
A	8	50	20	60	1200	400	1000	480
B	2	15	6	10	60	30	90	20
C	1	20	2	25	50	20	40	25
D	2	10	5	8	40	20	50	16
E	1	40	5	30	150	40	200	30
					1500	510	1380	571

$$po_1 = \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} x \frac{\sum p_1 q_1}{\sum p_0 q_1}} x100$$

$\sum p_1 q_0 = 1380$, $\sum p_0 q_0 = 510$, $\sum p_1 q_1 = 1500$, $\sum p_0 q_1 = 571$

$$po_1 = \sqrt{\frac{1380}{510} x \frac{1500}{571}} x100 = 266$$

Illustration -14: Construct Fisher's Ideal Index Number from the following data :

Commodity	Base Year Price	Base Year Quantity	Current Year Price	Current Year Quantity
Pencil	1	10	1.5	8
Pen	5	12	6.0	10
Scale	8	5	10.0	2

Solution :

Commodity	Base Year Price	Base Year Quantity	Current Year Price	Current Year Quantity		P_1q_1	P_0q_0	P_1q_0	P_0q_1
Pencil	1	10	1.5	8	15	10	12	8	
Pen	5	12	6.0	10	72	60	60	50	
Scale	8	5	10.0	2	50	40	20	16	
					137	110	92	74	

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$$

$\sum p_1q_0 = 137$, $\sum p_0q_0 = 110$, $\sum p_1q_1 = 92$, $\sum p_0q_1 = 74$

$$po_1 = \sqrt{\frac{137}{110} x \frac{92}{74}} x100$$

$$po_1 = \sqrt{1.245x1.243}x100 = 124.399$$

Illustration-15 : From the following information construct Fisher's Ideal Index Number :

Commodity	P_0	P_1	q_0	q_1
Wheat	30	35	5	4
Rice	32	37	7	5
Dal	20	18	3	4

Solution :

Commodity	P_0	P_1	q_0	q_1	P_1q_1	P_0q_0	P_1q_0	P_0q_1
Wheat	30	35	5	4	140	150	175	120
Rice	32	37	7	5	185	224	259	160
Dal	20	18	3	4	72	60	54	80
				397	434	488	360	

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$$

$\sum p_1q_0$ = 488, $\sum p_0q_0$ = 434, $\sum p_1q_1$ = 397, $\sum p_0q_1$ = 360

$$po_1 = \sqrt{\frac{488}{434} x \frac{397}{360}} x100$$

$$po_1 = \sqrt{1.124 x 1.102} x100$$

$$po_1 = \sqrt{1.238} x100 = 111.26$$

Illustration -16: From the following data construct Fisher's Ideal Index Number.

Item	Base Year		Current Year	
	Unit Price	Expenditure	Unit Price	Expenditure
Chocklet	2	40	5	75
Biscuit	4	16	8	40
Lollypup	2	20	2	24
Cadbury	5	25	10	60
Milky Bar	3	18	6	30

Solution : Here quantity is not given. Hence quantity is calculated as

$$= \frac{\text{Expenditure}}{\text{Unit Price}}$$

Item	Base Year Quantity(q0)	Current Year Quantity(q1)
Chocklet	$\frac{40}{2} = 20$	$\frac{75}{5} = 15$
Biscuit	$\frac{16}{4} = 4$	$\frac{40}{8} = 5$
Lollypup	$\frac{20}{2} = 10$	$\frac{24}{2} = 12$

Cadbury	$\frac{25}{5} = 5$	$\frac{60}{10} = 6$
Milky Bar	$\frac{18}{3} = 6$	$\frac{30}{6} = 5$

Commodity	Base Year		Current Year							
	Price P_0	Quantity q_0	Price P_1	Quantity q_1		P_1q_1	P_0q_0	P_1q_0	P_0q_1	
Chocklet	2		20	5	15		100	40	75	30
Biscuit	4	4	8	5		32	16	40	20	
Lollypup	2	10	2	12		20	20	24	24	
Cadbury	5	5	10	6		50	25	60	30	
Milky Bar	3	6	6	5		36	18	30	15	
						238	119	229	119	

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$$

$\sum p_1q_0$ = 238, $\sum p_0q_0$ = 119, $\sum p_1q_1$ = 229, $\sum p_0q_1$ = 119

$$po_1 = \sqrt{\frac{238}{119} x \frac{229}{119}} x100$$

$$po_1 = \sqrt{2 x 1.924} \; x100$$

$$po_1 = \sqrt{3.848} \; x100$$

$P0_1$ = 1.962 X 100 = 196.2

Illustration - 17 : From the following information construct Fisher's Ideal Index Number and Laspeyer, Paasche, Bowley Index Numbers :

Item	2005		2006	
	Price	Quantity	Price	Quantity
A	4	6	2	8
B	6	5	5	10

C	5	10	4	14
D	2	13	2	19

Solution :

Commodity	Price P_0	Quantity P_1	Price q_0	Quantity q_1	P_1Q_1	P_0Q_0	P_1Q_0	P_0Q_1
A	4	6	2	8	12	24	16	32
B	6	5	5	10	25	30	50	60
C	5	10	4	14	40	50	56	70
D	2	13	2	19	26	26	38	38
					103	130	160	200

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$$

$\sum p_1q_0$ = 103 , $\sum p_0q_0$ = 130, $\sum p_1q_1$ = 160, $\sum p_0q_1$ = 200

$$po_1 = \sqrt{\frac{103}{130} x \frac{160}{200}} x100$$

$$po_1 = \sqrt{0.7923\, x 0.8} x100$$

$$po_1 = \sqrt{0.6336}\, x100$$

$P0_1$ = 0.796 X 100 = 79.6

Fisher Index Number = 79.6

Paasche's Index Number = $\frac{\sum p_1q_0}{\sum p_0q_0} x100$

$= \frac{160}{200} x100$

= 0.7923 x 100 = 79.23

Laspeyer's Index Number = $\frac{\sum p_1q_1}{\sum p_0q_1} x100$

$= \frac{103}{130} x100$

= 0.8 x 100 = 80

Bowley's Index Number = $\dfrac{\dfrac{\sum p_1q_0}{\sum p_0q_0}+\dfrac{\sum p_1q_1}{\sum p_0q_1}}{2} x100$

= $\dfrac{\dfrac{103}{130}+\dfrac{160}{200}}{2} x100$

= $\dfrac{0.7923+0.8}{2} x100$

= $\dfrac{1.592}{2} x100$

= 0.796 x 100

= 79.6

Illustration -18: From the following data construct Index Numbers by using Laspeye's method, Paasche Method, Bowley, Marshall - Edgeworth and Fisher's Ideal Index numbers.

	2005		2006	
Commodity	P_0	q_0	P_1	q_1
P	20	8	40	6
Q	50	10	60	5
R	40	15	50	15
S	10	20	20	25

Solution :

Commodity	P_0	q_0	P_1	q_1	P_1q_1	P_0q_0	P_1q_0	P_0q_1
P	20	8	40	6	320	160	240	120
Q	50	10	60	5	600	500	300	250
R	40	15	50	15	750	600	750	600
S	10	20	20	25	400	200	500	250
				2070	1460	1790	1220	

1. Fisher's Ideal Index $po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$

$$po_1 = \sqrt{\frac{2070}{1460} x \frac{1790}{1220}} x100$$

$$po_1 = \sqrt{1.417 x 1.4672} x100$$

$$po_1 = \sqrt{2.0802}\ x100$$

$P0_1$ = 1.442 X 100 = 144.2

Bowley's Index Number = $\frac{\frac{\sum p_1q_0}{\sum p_0q_0} + \frac{\sum p_1q_1}{\sum p_0q_1}}{2} x100$

$$= \frac{\frac{2070}{1462} + \frac{1790}{1220}}{2} x100$$

$$= \frac{1.4178 + 1.4672}{2} x100$$

$$= \frac{2.8850}{2} x100$$

= 1.4425 x 100

= 144.25

Laspeyre's Index Number = $\frac{\sum p_1q_0}{\sum p_0q_0} x100$

$$= \frac{2070}{1462} x100$$

= 1.4178 x 100

= 141.78

Paasche's Index Number = $\frac{\sum p_1q_1}{\sum p_0q_1} x100$

$= \frac{1790}{1220} x100$

= 1.4672 x 100

= 146.72

Marshall-Edgeworth Index Number = $\frac{\sum p_1q_0 + \sum p_1q_1}{\sum p_0q_0 + \sum p_0q_1} x100$

$= \frac{2070 + 1790}{1462 + 1220} x100$

$= \frac{3860}{2680} \ x \ 100$

= 1.440298 x 100

= 144.02

Illustration-19 :

From the following data construct Weighted Index numbers of Fisher's, Marshall-Edgeworth and Kelley.

	Base Year		Current Year	
Item	Quantity	Total	Quantity	Total
A	20	40	45	180
B	24	96	30	150
C	30	180	40	320
D	40	320	60	600

Solution:

Here Price are not given.

Price = $\frac{\text{Total}}{\text{Quantity}}$

Base year Prices	Current year Prices
40/20 = 2	180/45=4
96/24 = 4	150/30 = 5
180/30 = 6	320/40 = 8
320/40 = 8	600/60 = 10

Commodity	Base Year Price P_0	Base Year Quantity q_0	Current Year Price P_1	Current Year Quantity q_1	P_1q_1	P_0q_0	P_1q_0	P_0q_1
A	2	20	4	45	80	40	180	90
B	4	24	5	30	120	96	150	120
C	6	30	8	40	240	180	320	240
D	8	40	10	60	400	320	600	480
					840	636	1250	930

	$\frac{q_1+q_0}{2} xq$		P_1q	P_0q
A	3.5		130	65
B	27		135	108
C	35		280	210
D	50		500	400
		1045	783	

	P_1q	P_0q
A	q_1 - 45	
	q_0 - $\frac{\frac{20}{65}}{2}$ =32.5 X 4 = 130	2 X 32.5 = 65
B	q_1 - 30	
	q_0 - $\frac{\frac{24}{54}}{2}$ =27 X 5 = 135	4 X 27 = 108
C	q_1 - 40	
	q_0 - $\frac{\frac{30}{70}}{2}$ =35 X 8 = 280	6 X 35 = 210
D	q_1 - 60	
	q_0 - $\frac{\frac{40}{100}}{2}$ =50 X 10 = 500	8 X 50 = 400

Fisher's Ideal Index $po_1 = \sqrt{\frac{\sum p_1 q_0}{\sum p_0 q_0} x \frac{\sum p_1 q_1}{\sum p_0 q_1}} x100$

$\sum p_1 q_0$ = 840, $\sum p_0 q_0$ = 636, $\sum p_1 q_1$ = 1250, $\sum p_0 q_1$ = 930

$$po_1 = \sqrt{\frac{840}{636} x \frac{1250}{930}} x100$$

$$po_1 = \sqrt{1.32 X 1.34} x100$$

$$po_1 = \sqrt{1.7688} x100$$

$P0_1$ = 1.33 X 100 = 138

Fisher's Ideal Index Number = 138

Marshall-Edgeworth Index Number = $\frac{\sum p_1 q_0 + \sum p_1 q_1}{\sum p_0 q_0 + \sum p_0 q_1} x100$

$$= \frac{840 + 1250}{636 + 930} x100$$

$$= \frac{2090}{1566} x\,100$$

= 1.3346 x 100

= 133.46

Marshall-Edgeworth Index Number = 133.46

Kelley Index Number = $\frac{\sum p_1 q}{\sum p_0 q} x100$

= $\frac{1045}{783} x100$ = 1.334 x 100 = 133.46

Kelley Index Number = 133.46

Illustration-20 :

From the following information calculate Fisher's Index Number and Simple aggregative Index Number.

	Base Year		Current Year	
Item	Price	Expenditure	Price	Expenditure
1	2	40	5	75
2	4	16	8	40
3	1	10	2	24
4	5	25	10	60

Solution : Here quantity is not given so, Quantity = $\frac{\text{Expenditure}}{\text{Price}}$

	Base Year		Current Year					
	Price	Expenditure	Price	Expenditure				
Item	P_0	Q_0	P_1	q_1	P_1q_1	P_0q_0	P_1q_0	P_0q_1
1	2	40/2 = 20	5	75/5 = 15	100	40	75	30
2	4	16/4 = 4	8	40/8 = 5	32	16	40	20
3	1	10/1 = 10	2	24/2 = 12	20	10	24	12
4	5	25/5 = 5	10	60/10 = 6	50	25	60	30
	12		25		202	91	199	92

Simple Aggregative Method :

Index Number = $\frac{\sum P_1}{\sum P_0} x100$

$\sum P_1$ = 25, $\sum P_0$ = 12

= $\frac{25}{12} x100$ = 2.083 x 100 = 208.3

$P0_1$ = 208.3

Fisher's Ideal Index $po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$

$\sqrt{\frac{202}{91} x \frac{199}{92}} x100$

$\sqrt{2.222x2.16}x100$

$\sqrt{4.7952}x100$

= 2.19 X 100 = 219

Fisher's Ideal Index Number = 219

Illustration-21 : From the following data construct Index Numbers of Fisher's, Bowley, Marshal-Edgeworth.

Commodity	Quantity		Value	
	2005	2006	2005	2006
P	100	150	500	900
Q	80	100	320	500
R	60	72	150	360
S	30	33	360	297

Solution : Here price is not given, Price = $\frac{\text{Value}}{\text{Quantity}}$

	2005		2006					
Commodity	P_0	Q_0	P_1	Q_1	P_1q_1	P_0q_0	P_1q_0	P_0q_1
A	500/100=5	100	900/150=6	150	600	500	900	750
B	320/80=4	80	500/100=5	100	400	320	500	400
C	150/60=2.5	60	360/72=5	72	300	150	360	180
D	360/30=12	30	297/33=9	33	270	360	297	396
					1570	1330	2057	1726

Fisher's Ideal Index $po_1 = \sqrt{\frac{\sum p_1q_o}{\sum p_0q_o} x \frac{\sum p_1q_1}{\sum p_0q_1}}x100$

$\sqrt{\frac{1570}{1330} x \frac{2057}{1726}}x100$

$\sqrt{1.18X1.19}x100$

$\sqrt{1.4042}x100$

= 1.185 X 100 = 118.5

Fisher's Ideal Index Number = 118.5

Bowley's Index Number = $\dfrac{\dfrac{\sum p_1q_0}{\sum p_0q_0}+\dfrac{\sum p_1q_1}{\sum p_0q_1}}{2} x100$

$= \dfrac{\dfrac{1570}{1330}+\dfrac{2057}{1726}}{2} x100$

$= \dfrac{1.18+1.19}{2} x100$

$= \dfrac{2.37}{2} x100$

= 1.185 x 100

= 118.5

Marshall-Edgeworth Index Number = $\dfrac{\sum p_1q_0+\sum p_1q_1}{\sum p_0q_0+\sum p_0q_1} x100$

$= \dfrac{1570+2057}{1330+1726} x100$

$= \dfrac{3627}{3056} x\,100$

= 1.1868 x 100

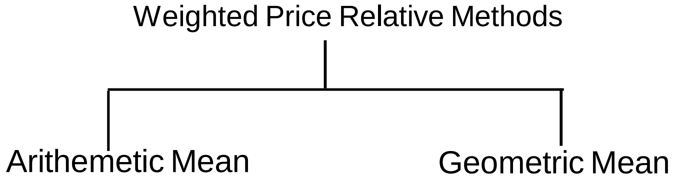
= 118.68

7.5.2 Weighted Average of Price Relatives or Family Budget Method or Cost of Living Index Number:

This method is popularly known as the Weighted price relaive method or Family budget method or cost of living index number under this method index number is computed in two ways -

1. Arithmetic Mean (A.M.)

2. Geometric Mean (G.M.)



a) **Index Number by using Arithmetic Mean**

The steps in the computation of the Weighted Arithmetic Mean of relatives Index Number are as follows:

i) Express each item of the period for which the index number is being calculated as a percentage of the same item in the base period.

ii) Multiply the percentages as obtained in step (i) for each by the weight which hasbeen assigned to that item.

iii) Add the results obtained.

iv) Divide the sum obtained by the sum of the weights usd. The result in the index number. Symbolically

$$P_{01} = \frac{\sum IV}{\sum V}$$

I = Price relatives $P = \frac{P_1}{P_0} x100$

V = Value/ Weights i.e. $P_0 q_0$

b) **Index number by using Geometric Mean :**

The Geometric mean may be used for averaging relatives. When this method is used the formula for computing the index is

$$P_{01} = A \log \frac{\sum I \log V}{\sum V}$$

A.log = Anti log

$$I = \frac{P_1}{P_0} x100$$

V = $P_0 q_0$ (If weights are not given)

log I = Logarithms of relatives

V log I = V x log

$\sum$ V = Total of Weights

Illustration-22 : From the following data construct Index Number Family Budget Method by using Arithmetic Mean.

Expenditure	Food	Rent	Cloth	Fuel	Others
	40%	10%	20%	15%	15%
Prices in 2005	80	20	80	10	50
Prices in 2006	100	30	90	20	60

Solution :

Expenditure	Weights (V)	Prices in 2005	Prices in 2006	$I = \frac{P_1}{P_0} x100$	IV
		P_0	P_1		
Food	40	80	100	125	5000
Rent	10	20	30	150	1500
Cloth	20	20	90	113.5	2250
Fuel	15	10	20	200	3000
Others	15	50	60	120	1800
	100				13550

$$P01 = \frac{\sum IV}{\sum V}$$

$\sum$ IV = 13550

$\sum$ V = 100

$$P_{01} = \frac{13550}{100} = 135.50$$

Illustration-23 : From the following data calculate Index Numbe under Family Budget Method by using Arithmetic Mean.

	Base Year		Current Year	
Item	Price	Quantity	Price	Quantity
A	8	90	10	130
B	10	100	12	120
C	20	150	25	30
D	12	60	15	80

Solution :

	Base Year		Current Year				
Item	Price	Quantity	Price	Quantity	I	V	IV
	P_0	q_0	P_1	q_1	$I = \frac{P_1}{P_0} x100$	P_0q_0	
A	8	90	10	130	125	720	90000
B	10	100	12	120	120	1000	120000
C	20	150	25	30	125	3000	375000

D	12	60	15	80	125	720	90000
						5440	675000

$$P_{01} = \frac{\sum IV}{\sum V}$$

$\sum IV = 675000$

$\sum V = 5440$

$$P_{01} = \frac{675000}{5440} = 124.08$$

Illustration-24 : Construct Cost of Living Index Number.

Item	Index Number	Weights
Food	3520	480
Lighting	2200	100
Cloth	2300	80
Rent	1600	120
Others	1900	150

Solution :

Item	Index Number (I)	Weights (V)	IV
Food	3520	480	1689600
Lighting	2200	100	220000
Cloth	2300	80	184000
Rent	1600	120	192000
Others	1900	150	285000
		930	2570600

$$P_{01} = \frac{\sum IV}{\sum V}$$

$\sum IV = 2570600$

$\sum V = 930$

$$P_{01} = \frac{2570600}{930} = 2763$$

Illustration-25 : From the following data construct Price relative Index Number by using Geometric Mean.

	Base Year		Current Year
Commodity	Price	Quantity	Price
P	3	30	4
Q	4	50	5
R	3	40	4
S	2	20	3
T	5	10	8

Solution :

	Base Year		Current Year				
Commodity	Price	Quantity	Price				
	P_0	q_0	P_1	$V=P_0q_0$	$I = \frac{P_1}{P_0} x100$	I log x	I log V
P	3	30	4	90	133.33	2.1249	191.241
Q	4	50	5	200	125	2.0969	419.380
R	3	40	4	120	133.33	2.1249	254.989
S	2	20	3	40	150	2.1761	82.044
T	5	10	8	50	160	2.2041	110.205
					500		1062.858

$$P_{01} = A\log\frac{\sum I \log V}{\sum V}$$

$$P_{01} = A\log\frac{1062858}{500}$$ = A log of 2.1257 = 133.6

7.6 QUANTITY OR VOLUME INDEX NUMBERS

Quanity or Volume Index Number is one that measures th changes in the level of quantities consumed during a given period with refrence to a base period. Just like the price indices the quantit indices can be constructed under he various methods. The only thing to do here is to reverse the position o p and q betwen themselves. In such index number, the prices are treated as weights and the quantities are treated as the variables. Following are the principles to construct quantity index numbers under different methods.

1. Laspeyre's Index Q_{01} $= \frac{\sum q_1 p_0}{\sum q_0 p_0} x100$

2. Paasche's Index q_{01} $= \frac{\sum q_1 p_1}{\sum q_0 p_0} x100$

3. Bowley's Index $q_{01} = \frac{\frac{\sum q_1 p_0}{\sum q_0 p_0} + \frac{\sum q_1 p_1}{\sum q_0 p_0}}{2} x100$

4. Marshall-Edgeworth Index $q_{01} = \frac{\sum q_1 p_0 + \sum q_1 p_1}{\sum q_0 p_0 + \sum q_0 p_1} x100$

5. Fisher's Ideal Index q_{01} $\sqrt{\frac{\sum q_1 p_0}{\sum q_0 p_0} x \frac{\sum q_1 p_1}{\sum q_0 p_1}} x100$

Illustration-26 : From the following data construct quantity index number by using Laspeyre's, Paasche, Bowly, Fisher and Marshall-Edgeworth principles.

	2005		2006	
Commodity	Price	Quantity	Price	Quantity
G	4	40	5	50
H	3	80	4	100
I	5	50	8	80
J	8	40	10	50
K	10	30	12	40

Solution :

	2005		2006					
Commodity	Price	Quantity	Price	Quantity				
	P_0	q_0	P_1	q_1	P_1q_1	P_0q_0	P_1q_0	P_0q_1
G	4	40	5	50	200	160	250	200
H	3	80	4	100	300	240	400	320
I	5	50	8	80	400	250	640	400
J	8	40	10	50	400	320	500	400
K	10	30	12	40	400	300	480	360
					1700	1270	2270	1680

$\sum q_1 p_0$ = 1700, $\sum q_0 p_0$ = 1270, $\sum q_1 p_1$ = 2270, $\sum q_0 p_0$ = 1680

Laspeyer's Index Number $q_{01} = \frac{\sum q_1 p_0}{\sum q_0 p_0} x100$

$= \frac{1700}{1270} x100$

= 1.338 x 100

= 133.8

Paasche's Index Number $q_{01} = \frac{\sum q_1 p_1}{\sum q_0 p_0} x100$

$= \frac{2270}{1680} x100$

= 1.351 x 100

= 135.1

Bowley's Index Number $q_{01} = \frac{\frac{\sum q_1 p_0}{\sum q_0 p_0} + \frac{\sum q_1 p_1}{\sum q_0 p_0}}{2} x100$

$= \frac{\frac{1700}{1270} + \frac{2270}{1680}}{2} x100$

$= \frac{1.338+1.351}{2} x100$

$= \frac{2.689}{2} x100$

= 1.3445 x 100

= 134.45

Fisher's Ideal Index $q_{01} = \sqrt{\frac{\sum q_1 p_0}{\sum q_0 p_0} x \frac{\sum q_1 p_1}{\sum q_0 p_1}} x100$

$\sqrt{\frac{1700}{1270} x \frac{2270}{1680}} x100$

$\sqrt{1.338 x 1.351} x100$

$\sqrt{1.807638} x100$

q_{01} = 1.3445 X 100 = 134.45

Marshall-Edgeworth Index Number $q_{01} = \frac{\sum q_1 p_0 + \sum q_1 p_1}{\sum q_0 p_0 + \sum q_0 p_1} x100$

$= \frac{1700 + 2270}{1270 + 1680} x100$

$= \frac{3970}{2960} x\,100$

= 1.3445 x 100

= 134.45

7.7 EXERCISE

1. What are the various problems faced by a peson while constructing an Index Number.
2. Describe the method of constructing an Index Number.

Problems on Simpel Aggregative Method and Simple Average of Relatives Method.

3. Construct simple aggregative price index from the following data.

Commodity	Prices in 2005	Prices in 2006
	Rs.	Rs.
A	15.00	18.00
B	5.00	8.00
C	2.25	4.00
D	6.00	7.00

(Ans.: 130.97)

4. From the following data construct Index Numbers by using Simple Aggregative Method and Simple aver
age of Relatives Method.

Commodity	P	Q	R	S	T	U
Prices in 2005	40	60	70	20	50	100
Prices in 2006	50	70	90	30	80	110

(Ans. : 123.53, 129.33)

5. From the following data construct index number by Simple Aggregative Method.

Commodity	A	B	C	D
Prices in 2005	162	256	257	132
Prices in 2006	171	164	189	145

(Ans. : 82.9)

6. Construct Index Number by Simple Aggregative Method.

Commodity	A	B	C	D	E	
Prices in 2005	10	20	4	5	10	
Prices in 2006	15	25	6	8	15	(Ans.: 140.82)

7. From the following data construct Index Numbers by Simple Aggregative Method and Simple Average of
Relatives Method.

Item	Bricks	Sand	Wood	Cement	Others	
Prices in 2005	50	80	105	120	5	
Prices in 2006	60	120	100	160	10	(Ans.; 125, 139.7)

INDEX NUMBERS THROUGH ARITHMETIC MEAN AND GEOMETRIC MEAN

8. From the following data construct Index Numbers by using Arithmetic mean and Geometric mean.

Commodity	A	B	C	D	E	
Prices in 2005	20	30	40	50	10	
Prices in 2006	30	60	60	80	30	(Ans.: 192)

9. From the following data construct Index Numbers by using Arithmetic mea and Geometric mean.

Commodity	A	B	C	D	E
Prices in 2005	5	12	2	10	6
Prices in 2006	5	8	4	15	8

WEIGHTED AGGREGATIVE INDEX NUMBERS

10. From the following data calculate Laspeyer's, Paasche, Bowley, Fisher's , Marshall-Edgeworth Index Numbers.

Commodity	Base Year		Current Year	
	P_0	q_0	P_1	q_1
A	2	20	3	40
B	5	40	6	80
C	4	80	8	100
D	8	90	10	120
E	10	20	12	40

(Ans.: 140.54, 137.5, 139.02, 139.02, 138.7)

11. From the following data find the weighted index numbers using.

1) Laspeyer's Method, 2) Paasche's Method, 3) Dorbish and Bowley's Method, 4) Fisher's ideal Method,

Marshal-Edgeworth Method and Kelley's Method.

Item	Base Year		Current Year	
	P_0	q_0	P_1	q_1
P	10	49	12	50
Q	12	25	15	20
R	18	10	20	12
S	20	5	40	2

(Ans.: 127.38, 122.49, 124.935, 124.9, 125.02, 125.02)

12. From the following data compute the quantity index number using Paasche's formula, Fisher's formula,

Bowley's formula and Marshall's formula.

Item	Quantity in Units		Value in Rupees	
	2005	2006	2005	2006
M	90	120	450	600
N	80	90	480	630
O	75	80	300	240
P	50	60	350	300

(Ans. : 119.19, 119.09, 119.09, 119.09)

13. Construct Fisher's Ideal Index.

Commodity	Base Year		Current Year	
	Price	Quantity	Price	Quantity
A	6	50	10	60
B	2	100	2	120
C	4	60	2	60
D	10	30	12	30
E	15	60	20	60

(Ans.: 122.71)

14. From the following data construct Laspeyer's, Paasche's, Fisher's Index Numbers.

	Prices		Quantity	
Commodity	2005	2006	2005	2006
A	5	5	15	5
B	7	4	5	3

C	8	6	6	10	
D	3	3	8	4	(Ans.: 85.17, 86.23, 85.5)

15. Calculate Fisher's Ideal Index Number.

	Quantity		Prices		
Commodity	2005	2006	2005	2006	
P	12	15	10	12	
Q	15	20	7	5	
R	24	20	5	9	
S	5	5	16	14	(Ans.: 115.76)

16. From the following data calculate Fisher's Ideal Index Number.

	Base Year		Current Year	
Item	Price	Total Value	Price	Total Value
Milk	8	80	10	110
Ghee	10	90	12	108
Oil	16	256	20	340

17. Calculate Fisher's Ideal Index Number.

	Base Year		Current Year		
Commodity	Price	Expenditure	Price	Expenditure	
Rice	10	120	12	144	
Maize	5	40	6	54	
Wheat	20	60	25	100	
Bazra	8	80	8	72	(Ans.: 116.49)

18. From the following data, calculate Laspeyre's and Paasche's Dorbish and Bowley, Marshall-Edgeworth
and Fisher's Index Number.

Item	Base Year		Current Year	
	Price (Rs.)	Quantity (Kg.)	Price (Rs.)	Quantity (Kg.)
A	6	50	10	56
B	2	100	2	120
C	4	60	6	60
D	10	30	12	24
E	8	40	12	36

(Ans.: 139.71, 139.88, 139.8, 139.79, 139.8)

19. Calculate Fisher's Ideal Index Number.

$\sum p_1q_0 = 122, \quad \sum p_1q_1 = 217$

$\sum p_0q_0 = 184 \quad \sum p_0q_1 = 190$ (Ans.: 114.71)

20. Calculate Index numbers from the following data by Laspeyre's, Paasche's, Bowley's and Fisher's method.

Commodity	P_0	Q_0	P_1	Q_1
I	8.0	5	10	11
II	8.5	6	9	9
III	9.0	4	12	6

(Ans. 119.69, 120.37, 120.03, 120.03)

WEIGHTED AVERAGE OF PRICE RELATIVE METHOD

21. Calculate Weighted Average of Price Relative Index.

Item	Unit	Quantity	Prices in 2005	Prices in 2006
Cement	100 pounds	50 pounds	5	8
Wood	S.F.	2,000 S.F.	9.50	1420
Iron	S.F.	50 S.F.	34	42
Bricks	per 1,000	20,000	12	24

22. Calculate Index Numbers by using Weighted Average of Price Relative Method.

	2005		2006		
Commodity	P_0	Q_0	P_1	Q_1	
Food	100	30	90	25	
Rent	20	15	20	20	
Clothes		70	20	60	30
Fuel	20	10	15	15	
Misc.	40	25	55	10	

(Ans.: 91.67)

QUANTITY INDEX NUMBER

23. Calculate Quantity Index Numbers from the following data by using Laspeyres, Paasche, Bowley and
Fisher's Methods.

	2005		2006	
Item	P_0	q_0	P_1	q_1
A	4	10	8	8

B	6	8	12	6
C	2	16	4	14
D	8	10	16	8

24. From the following data construct Quantity Index Numbers of Fisher, Laspeyre, Paasche, Bowley.

	Prices		Quantities	
Commodity	2005	2006	2005	2006
A	9.2	15	5	5
B	8	12	10	11
C	4	5	6	6
D	1	1.25	4	8

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Lesson 8

TESTS OF ADEQUACY OF INDEX NUMBER FORMULA

8.0 OBJECTIVE

After studying this lesson you should be able to understand :

1. Tests of Index numbers.
2. Chain Index numbers.
3. Fixed Index numbers.

STRUCTURE OF LESSON

8.1 Introduction

8.2 Tests of principles of Index Numbers

- **8.2.1 Unit Test**
- **8.2.2 Time Reversal Test**
- **8.2.3 Factor Reversal Test**
- **8.2.4 Circular Test**

8.3 Chain Indices

- **8.3.1 Fixed Base System**
- **8.3.2 Chain Base System**
- **8.3.3. Conversion of Chain Index into Fixed Base Index**
- **8.3.4 Base Shifting**

8.4 Exercises

8.1 INTRODUCTION

Several formulae have been suggested for constructing index numbers and the problem is that of selecting the most appropriate one in a given situation. An appropriate Index number can be said an ideal one. An index number will be an ideal only if it satisfies tests like -

1. Unit Test
2. Time Reversal Test
3. Factor Reversal Test
4. Circular Test

8.2 TESTS OF PRINCIPLES OF INDEX NUMBERS

8.2.1 Unit Test :The unit test requires that the formula for constructing an index should be independent of the units in which, or for which, prices and quantities are quoted. Except for the simple unweighted aggregative index all other formulae satisfy this test.

8.2.2 Time Reversal Test (TRT) : Time Reversal Test has been proposed by Prof. Irving Fisher. It is a test to determine whether a given method will work both ways in time, forward and backward, when the data for any two years are treated by the same method, but with the bases reversed, the two index numbers secured should be receiprocals of each other so that their product is unity. Symbolically, the following relation should be satisfied :

$Po_1 \times P1_0 = 1$

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}}$$

$$P1_0 = \sqrt{\frac{\sum p_0q_0}{\sum p_1q_0} x \frac{\sum p_0q_1}{\sum p_1q_1}}$$

Time Reversal Test (TRT) = $P1_0 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1} x \frac{\sum p_0q_0}{\sum p_1q_0} x \frac{\sum p_0q_1}{\sum p_1q_1}} = \sqrt{1}$ = 1

8.2.3 Factor Reversal Test (FRT) : This test has also been proposd by Prof. Irving Fisher. According to this test if the pice and quantity indices are computed for the same data, same base and current periods and using the same formula, then their product should give the true ratio, as price multiply by quantity gives total value. Symbolically

$Po_1 \times Q_{01} = \frac{\sum p_1q_1}{\sum p_0q_0}$

$$po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} \qquad q0_1 = \sqrt{\frac{\sum q_1p_0}{\sum q_0p_0} x \frac{\sum q_1p_1}{\sum q_0p_1}}$$

$$F.R.T. = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1} x \frac{\sum q_1p_0}{\sum q_0p_0} x \frac{\sum q_1p_1}{\sum q_0p_1}}$$

$$F.R.T. = \frac{\sum p_1q_1}{\sum p_0q_0}$$

8.2.3 Unit Test : The unit test requires that the formula for constructing an index should be independent of the units in which, or for which, prices and quantities are quoted. Except for the simpel unweighted aggregative index all other formulae satisfy this test.

8.2.4 Circular Test : Another test of the adequacy of index number formula is what is known as 'circular test'. This test enables us to adjust the index values from period to period without referring each time to the original base. A test of this shiftability of base is called the circular test.

Symbolically, if there are three indices Po_1 , P_{12} and P_{20} the circular test will be satisfied if $Po_1 \times P_{12} \times P_{20} = 1$

or

When the test is applied to the simple aggregative method, the result is

$$\frac{\sum P_1}{\sum P_0} x \frac{\sum P_2}{\sum P_1} x \frac{\sum P_0}{\sum P_2} = 1$$

When it is applid to fixed weight aggregative method, the result is

$$\frac{\sum p_1 q}{\sum p_0 q} x \frac{\sum p_2 q}{\sum p_1 q} x \frac{\sum p_0 q}{\sum p_2 q} = 1$$

An index which satisfies this test has the advantage of reducing the compuations every time a change in the base year has to be made, such index numbers can be adjusted from year to year without referring each time to the original bases.

Illustration :

Calculate from the following data the Fisher's Ideal Index and show how it satisfies the Time Reversal test and Factor Reversal test.

	Price		Quantity	
Item	2005	2006	2005	2006
A	8	20	50	60
B	2	6	15	10
C	1	2	20	25
D	2	5	10	8
E	1	5	40	30

Computation of Fisher's Ideal Index

Item	P_0	q_0	P_1	q_1	P_1q_0	P_0q_0	P_1q_1	P_0q_1
A	8	50	20	60	1000	400	1200	480
B	2	15	6	10	90	30	60	20
C	1	20	2	25	40	20	50	25
D	2	10	5	8	50	20	40	16
E	1	40	5	30	200	40	150	30
				1380	510	1500	571	

Fisher's Ideal index $Po_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1}} x100$

$\sum p_1q_0 = 1380$

$\sum p_0q_0 = 510$

$\sum p_1q_1 = 1500$

$\sum p_0q_1 = 571$

$Po_1 = \sqrt{\frac{1380}{510} x \frac{1500}{571}} x100$ = 2.661 x 100 = 266.61

$P0_1 = 266.61$

Time Reversal Test : Time Reversal test is satisfied when $Po_1 x P1_0 = 1$

$Po_1 x P1_0 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1} x \frac{\sum p_0q_0}{\sum p_1q_0} x \frac{\sum p_0q_1}{\sum p_1q_1}} = 1$

$\sqrt{\frac{1380}{510} x \frac{1500}{571} x \frac{510}{1380} x \frac{571}{1500}} = 1$

Hence Time Reversal Test is satisfied by the given data.

Factor Reversal Test : Factor Reversal Test is satisfied when $Po_1 x qo_1 = \frac{\sum p_1q_1}{\sum p_0q_0}$

$$Po_1 \times qo_1 = \sqrt{\frac{\sum p_1q_0}{\sum p_0q_0} x \frac{\sum p_1q_1}{\sum p_0q_1} x \frac{\sum q_1p_0}{\sum q_0p_0} x \frac{\sum q_1p_1}{\sum q_0p_1}} = \frac{\sum p_1q_1}{\sum p_0q_0}$$

$$Po_1 \times qo_1 = \sqrt{\frac{1380}{510} x \frac{1500}{571} x \frac{571}{510} x \frac{1500}{1380}} = \frac{1500}{510}$$

Hence Factor Reversal Test is satisfied by the given data.

8.3 DIFFERENT METHODS OF COMPUTING THE SERIES OF INDEX NUMBERS

As pointed out earlier, there are two methods of computing the series of Index numbers. They are

1. Fixed base method
2. Chain base method

8.3.1 Fixed Base Method : Under this method, the base year remains fixed for all the years of calculation. The formula for calculating the price indices under this method is as follows

$$Po_1 = \frac{P_1}{P_0} x 100$$

Po_1 = Price index of the current year on the basis of the price of a fixed base year.

P_1 = Price of the current year.

Po = Price of fixed base year.

8.3.2 Chain Base Method : Under this method, the base year's price does not remain fixed but moves step by step from year to year. In other words, the immediately preceding year's price becomes the base year's price for each of the succeeding years.

Steps in Construction of Chain Index Numbers

i. Express the figures for each year as percentages of the preceding year. The results so obtained are called link relatives.

ii. Chain together these percentages by successive multiplication to form a chain index. the formula is -

$$\text{Chain Index for current year} = \frac{\text{Average link relative of current year} \times \text{Chain Index of Previous Year}}{100}$$

Illustration

From the following data relating to the average prices of a commodity compute the index numbers for each of the ten years taking 1995 as the base year, and compute chain base Index numbers.

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Price	50	60	55	65	75	80	70	60	64	85

Solution

Computation of the Fixed Base Index Numbers or the Price Relatives for the Ten Years

Year	Price	Fixed Base Index Numbers 1995 - Base Year	
		Calculation $\frac{P_1}{P_0} \times 100$	Indices
1995	50	50/50 x 100	100
1996	60	60/50 x 100	120
1997	55	55/50 x 100	110
1998	65	65/50 x 100	130
1999	75	75/50 x 100	150
2000	80	80/50 x 100	160
2001	70	70/50 x 100	140
2002	60	60/50 x 100	120
2003	64	64/50 x 100	128
2004	85	85/50 x 100	170

Computation of the Chain Base Index Numbers of the Link Relatives for the Ten Years

Year	Price	Chain Base Index Numbers		Chain Index Number
		Calculations	Indices	
1995	50	50/50 x 100	100	100
1996	60	60/50 x 100	120	$\frac{120}{100} \times 100 = 120$
1997	55	55/60 x 100	91.67	$\frac{91.67}{120} \times 100 = 76.39$
1998	65	65/55 x 100	118.18	$\frac{118.18}{76.39} \times 100 = 154.7$
1999	75	75/65 x 100	115.38	$\frac{115.38}{154.7} \times 100 = 74.58$
2000	80	80/75 x 100	106.67	$\frac{106.67}{74.58} \times 100 = 143.03$
2001	70	70/80 x 100	87.50	$\frac{87.5}{143.03} \times 100 = 61.18$
2002	60	60/70 x 100	85.71	$\frac{85.71}{61.18} \times 100 = 140.1$
2003	64	64/60 x 100	106.67	$\frac{106.67}{140.1} \times 100 = 76.14$
2004	85	85/64 x 100	132.81	$\frac{132.81}{76.14} \times 100 = 174.43$

8.3.3 Conversion of Fixed Base Index, Chain Base Index Numbers

I) CONVERSION OF CHAIN INDEX NUMBERS INTO FIXED INDEX NUMBERS

For converting the chain base index numbers into the fixed base index numbers the following principle to be applied.

$$FBI = \frac{\text{Current CBI Previous FBI}}{100}$$

FBI = Fixed Base Index

CBI = Chain Base Index

Illustration :

Convert following Chain index Numbers into fixed Index Numbers.

Years	1990	1991	1992	1993	1994
Chain Index	80	110	120	90	140

Solution :

Year	Chain Index Numbers	Fixed Base Index Numbers
1990	80	100 x 80/ 100 = 80
1991	110	110 x 80 / 100 = 88
1992	120	120 x 88 / 100 = 105.6
1993	90	90 x 105.6 / 100 = 95.04
1994	140	140 x 95.04 / 100 = 133.1

II. CONVERSION OF FIXED BASE INDEX NUMBERS INTO CHAIN INDEX NUMBERS

For converting the fixed base index numbers into the chain base index numbers the following simple formula is to be applied

$$CBI = \frac{P_1}{P_0} x 100$$

CBI = Chain Base Index

Illustration :

Convert following Fixed Base Index Numbers into Chain Base Index Numbers

Year	2000	2001	2002	2003	2004	2005
Fixed Base Index	376	392	408	380	392	400

Solution :

Year	Fixed Base Index	Chain Base Index Numbers
2000	376	100
2001	392	392 / 376 x 100 = 104.26
2002	408	408 / 392 x 100 = 104.08
2003	380	380 / 408 x 100 = 93.14
2004	392	392 / 380 x 100 = 106.16
2005	400	400 / 392 x 100 = 102.04

Illustration :

From the following data relating to average prices of a commodity find -

1. Link relatives
2. Price relatives
3. Chain indices

Year	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004
Price	75	65	50	72	60	70	78	75	80	84

Solution :

Computation of the Link Realtives, Price Relatives and Chain Indices

Year	Price	Link Relatives	Price Relatives	Chain Indices
1995	75	75 / 75 x100=100	75 / 75 x100=100	100 x 100 / 100=100
1996	65	65 / 75 x100=87	65 / 75 x100=87	87 x 100 / 100=87
1997	50	50 / 65 x100=77	50 / 75 x100=67	77 x 87 / 100 = 67
1998	72	72 / 50 x100=144	72 / 75 x100=96	144 x 67 / 100 = 96
1999	60	60 / 72 x100=83	60 / 75 x100=80	83 x 96 / 100 = 80
2000	70	70 / 60 x100=117	70 / 75 x100=93	117 x 80 / 100 = 93
2001	78	78 / 70 x100=111	78 / 75 x100=104	111 x 93 / 100 = 104
2002	75	75 / 78 x100=96	75 / 75 x100=100	99 x 104 / 100 = 100
2003	80	80 / 75 x100=107	80/ 75 x100=107	107 x 100 / 100 = 107
2004	84	84 / 80 x100=105	84 / 75 x100=112	105 x 107 / 100 = 112

Note : Approximated to the Unit Places.

8.3.4 Base Shifting : Base shifting means shifting of one fixed base period to another fixed base period of a series of index numbers. Base shifting for a given series of index numbers can be done by any of the following two methods.

a) Direct Method of Revision

Under this method the entire series of index number is thoroughly recast with reference to the new base year's price thus fixed subsequently.

b) Short Cut Method of Conversion

Under this method the given index numbers are shortly converted by application of the following formula.

$$\text{Revised Index Number} = \frac{\text{Old Index Number} \times 100}{\text{Old Index of the New Base Year}}$$

Where, C stands for the Common factor

$$\text{i.e.} \quad \frac{100}{\text{Old Index of the new base year}}$$

In this method, old index of the new base years is taken as the fixed base for all the years of the series.

8.4 EXERCISES

1. Explain Factor Reversal Test, Time Reversal Test.

2. What is Chain Index.

3. What is meant by Base Shifting.

4. From the following data, calculate Fisher's Ideal Index number and Prove how it satisfies Time Reversal

Test and Factor Reversal Test.

Item	Base Year		Current Year	
	P_0	Q_0	P_0	Q_0
Wheat	6	50	8	100
Ghee	10	15	11	20
Fuel	2	20	4	30
Sugar	5	10	8	30
Cloth	10	40	12	50

(Ans. : 129.64)

5. From the data given below prove that the Fisher's ideal index number satisfies the time reversal test and factor reversal test.

Item	Base Year		Current Year	
	Price	Quanity	Price	Quantity
N	5	50	8	40
O	7	25	12	30
P	9	10	15	25
Q	12	5	20	18

6. From the following data calculate Index numbers through different methods, and show which method satisfies Time Reversal Test and Factor Reversal Test.

Item	2005		2006	
	Price	Expenditure	Price	Expenditure
A	40	480	48	576
B	20	160	24	316
C	80	240	100	400
D	32	320	32	288

7. From the following data construct Fisher's Ideal Index number and prove how it satisfies Time Revesal Test and Factor Reversal Test.

Commodity	2005		2006	
	Price	Total Value	Price	Total Value
A	8	80	10	110
B	10	90	12	108
C	16	256	20	340

8. From the following data construct Fisher's Ideal Index number and prove how it satisfy Time Revesal Test and Factor Reversal Test.

(Ans. : 132.05)

9. From the following data construct Fisher's Idea Index Number and Prove how it satisfies Time Revesal Test and Factor Reversal Test.

Commodity	2005		2006	
	Quantity	Total	Quantity	Total

A	40	320	50	450
B	80	320	90	360
C	60	300	70	420
D	20	180	20	320

10. From the following data construct Fixed Base Index Numbers and Chain Base Index.

Year	1998	1999	2000	2001	2002	2003	2004	2005	2006
Price	70	50	60	75	70	50	40	60	75

11. Construct Index Numbers by taking 1998 as Base Year.

Year	1998	1999	2001	2002	2003	2004	2005	2006
Price	76	99	79	92	78	88	98	91

12. From the following link relatives construct chain Index Numbers.

Year	2000	2001	2002	2003	2004	2005	2006
Link Relatives	100	120	110	125	130	115	95

13. From the following Chain Index Numbers construct Fixed Base Index Numbers.

Year	1996	1997	1998	1999	2000	2001	2002	2003
Chain Indices	90	110	120	115	130	120	150	140

14. From the following Chain Indices construct Fixed Base Index Numbers.

Year	1998	1999	2000	2001	2002	2003	2004
Chain Indices	160	220	240	210	250	260	270

15. Construct Fixed Base Index Numbers from Cahin Base Index Numbers.

Year	2000	2001	2002	2003	2004	2005
Chain Indices	210	150	142	210	200	180

16. Construct Chain Indices from Fixed Base Indices.

Year	2001	2002	2003	2004	2005	2006

Fixed Base Index	200	300	231.4	271	350	400

17. Construct Chain Indices from Fixed Base Index Numbers.

Year	1999	2000	2001	2002	2003	2004	2005	2006
Production of Paddy	128	122	116	120	120	130	135	146

18. From the following Fixed Base Index Numbers construct Chain Indices.

Year	2001	2002	2003	2004	2005	2006
Fixed Base Index	100	110	120	125	140	150

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Lesson-9

DEFLATING INDEX NUMBERS

9.0 OBJECTIVE

After studying this lesson you should be able to understand

1. What is deflation.

2. What are the methods of deflation

STRUCTURE OF LESSON

9.1 INTRODUCTION

The word deflating is the verbal form of the word 'deflation' which is just opposite to the word inflation. Thus deflating means counteracting the effect of inflation over a set of data to unravel their true values and make them, comparable.

9.2 DEFLATION

Since price level and their indices are very often subject to inflation, it becomes necessary to deflate them properly to determine the true and comparable value of certain related factors such as Realwage, Real income, purchase power etc.,By deflating we mean making allowances for the effect of changing price levels. A rise in price level means a reduction in the purchasing power of money. For instance, the price of paddy rises from Rs. 250 per quintal in 1996 to Rs.500 per quintal in 2006, it means that in 2006. One can buy only half of paddy if he spends the same amont which he was spending on wheat in 1996, or in other words, the value of rupee is only 50 paise in 2006 as compared to 1996. Since the value of money goes down with rising prices the workers or the salaried people are interested not so much in money wages as in real wages i.e. not how much they earn but how much their income or wage will buy.

9.3 METHODS OF DEFLATING

For calculating real wages we can multiply money wages by a quantity measuring the purchasing power of the rupee, or better we divide the cash wages by an appropriate price index. This process is referred to as deflating. Following are the methods useful to measure deflation.

9.3.1 Purchasing Power of Money : This can be deflated by the following formula.

$$\text{Purchasing power of money} = \frac{100}{\text{PriceIndex}}$$

Thus if price rises by 25%, the price index becomes 125 and in that case the purchasing power of every rupe would be 100/125 = 0.80 ps. This means that rupee one in the current year is equal to 0.80 ps in the base year.

9.3.2 Real Wage or Income : This can be deflated by the following formula.

$$\text{Real wages} = \frac{\text{Money Wage}}{\text{Price Index}} \times 100$$

Here, price index shold be preferably the consumer price index rather than the wholesale price index as the former reflects very well the changes in the purchasing power of a wage earner.

Thus if a worker earns Rs.1,500 during a year in which the price index stands at 150, the real value of his wage in comparison to the wage of the base year would be 1500 / 150 x 100 = Rs.1,000. This indicates that his present wage of Rs. 1,500 is equal to the wage of Rs. 1,000, earned in the base year.

9.3.3 Real Wage Index or Real Income Index : Real Wage Index which is also known as Real Income Index can be deflated by the following formula.

$$\text{Real Wage Index Number} = \frac{\text{Index of money wage}}{\text{Price Index number}} \times 100$$

or

$$= \frac{\text{Real Wage of the Current Year}}{\text{Real Wage of the Base Year}} \times 100$$

Illustration 1 :

Calculate Real Wages from the following wages and price indices

Year	Wages	Price Indices
2003	1800	100
2004	2200	170
2005	3400	300
2006	3600	320

Solution :

Year	Wages	Price Indices	Real Wages $\left(\frac{\text{Wage}}{\text{Price Index}} \text{x100}\right)$
2003	1800	100	$\frac{1800}{100}$ x 100 = 1800
2004	2200	170	$\frac{2200}{170}$ x 100 = 1294.1
2005	3400	300	$\frac{3400}{300}$ x 100 = 1133.3
2006	3600	320	$\frac{3600}{320}$ x 100 = 1125.0

Illustration - 2:

From the following data calculate Real Wage Indices

Year	Wages(Rs.)	Price Indices
2000	200	100
2001	240	160
2002	350	280
2003	360	290
2004	360	300
2005	370	320
2006	375	330

Solution :

Year	Wages	Price Indices	Real Wages $\left(\frac{\text{Wage}}{\text{Price Index}} \text{x100}\right)$	Real Wage Indices $\left(\frac{\text{Real wages of Current}}{\text{Real Wage of base year}} \text{x100}\right)$
2000	200	100	$\frac{200}{100}$ x 100 = 200	$\frac{200}{200}$x 100 = 100.00
2001	240	160	$\frac{240}{160}$ x 100 = 150	$\frac{150}{200}$x 100 = 75.00
2002	350	280	$\frac{350}{280}$ x 100 = 125	$\frac{125}{200}$x 100 = 62.50
2003	360	290	$\frac{360}{290}$ x100=124.14	$\frac{124.14}{200}$x 100 = 62.07
2004	360	300	$\frac{360}{300}$ x100=120	$\frac{120}{200}$x 100 =60.00
2005	370	320	$\frac{370}{320}$ x100=115.63	$\frac{115.63}{200}$x 100 =57.82
2006	375	330	$\frac{375}{330}$ x100=113.64	$\frac{113.64}{200}$x 100 = 56.82

Illustration - 3 :

From the following data relating to the annual wages and the price indices determine the following by deflation :

1. The purchasing power of money
2. Real Wages and
3. Real Wage Index.

Year :	1998	1999	2000	2001	2002	2003	2004
Wage (Rs.):	180	220	340	360	365	370	375
Price Indices :	100	170	300	320	330	340	350

Also, ascertain the amount of wages and the percentage increase needed in 2004 to provide a buying power equal to that enjoyed in 1998.

Solution :

a) Determination of the purchasing power of money, Real Wage and Real Wage Index by deflating the index numbers :

Year	Wages Rs	Price Index	Pruchasing power of money $\left(\frac{100}{\text{Price Index}}\right)$	Real Wages $\left(\frac{\text{Money Wages}}{\text{Price Index}} x100\right)$	Real Wage Indices $\left(\frac{\text{Real Wages of current year}}{\text{Real Wage of Base year}} x100\right)$
1998	180	100	$\frac{100}{100} = 1.00$	$\frac{180}{100}$ x 100 = 180	$\frac{180}{180}$ x 100 = 100
1999	220	170	$\frac{100}{170} = 0.59$	$\frac{220}{170}$ x 100 = 129.41	$\frac{129.41}{180}$ x 100 = 72
2000	340	300	$\frac{100}{300} = 0.33$	$\frac{340}{300}$ x 100 = 113.33	$\frac{113.33}{180}$ x 100 = 63
2001	360	320	$\frac{100}{320} = 0.31$	$\frac{360}{320}$ x 100 = 112.50	$\frac{112.5}{180}$ x 100 = 62.5
2002	365	330	$\frac{100}{330} = 0.30$	$\frac{365}{330}$ x 100 = 110.61	$\frac{110.61}{180}$ x 100 = 61.0
2003	370	340	$\frac{100}{340} = 0.29$	$\frac{370}{340}$ x 100 = 108.82	$\frac{108.82}{180}$ x 100 = 60
2004	375	350	$\frac{100}{350} = 0.29$	$\frac{375}{350}$ x 100 = 107.14	$\frac{107.14}{180}$ x 100 = 59.5

b) The amount of wages needed in 2004 w.r.t. 1998

$$= \frac{180}{100} \text{ x } 350 = 630$$

Thus if the worker is paid Rs.630 annually, he will be able to enjoy the same purchasing powr as he was enjoying in 1998.

c) The Percentage of Increase in Wages of 2004

Wages needd in 2004 = 630

Less Wages received in 2004 = 375

Wages to be increased = 255

The percentage of increase = $\frac{255}{375} \times 100 = 68\%$

Therefore it needs to increase his wage of 2004 by 68% in order that it may be equal to the wage of 1998.

Illustration - 4:

During a certain period the consumer price index went up from 150 to 250 and the salary of a worker raised from Rs.450 to Rs.550. State by how much the worker has gained or lost both in money wage and in real wage ?

Solution :

i) In terms of money wage

Money wage = Slary received

When index was 150, Salary was 450.

When index is 250 salary should be

$\frac{450}{150} \times 250 = 750$

But his salary has gone only upto Rs.550. Thus, the worker has lost in terms of money wages by Rs.200 (750 - 550)

ii) In terms of real wage.

Real Wage = $\frac{\text{Money Wage}}{\text{Price Index}} \times 100$

Real Wage in the first year = $\frac{450}{100} \times 150 = 300$

Real Wage in the second year = $\frac{550}{250} \times 100 = 220$

Thus, in terms of real wage also, the worker has lost by Rs.80 (300 - 220).

9.4 LIMITATION OF DEFLATION

In principle deflation appears to be very simple but in practice the main difficulty consists in finding appropriate index to deflate a given set of values or appropriate deflators.

9.5 EXERCISE

1. What is meant by Deflation.

2. What are the uses of Deflation.

3. From the following data calculate Real Incomes.

Year :	2000	2001	2002	2003	2004	2005	2006
Wage Indices :	100	120	175	180	180	185	187.5
Price Indices :	100	160	280	290	300	320	330

(Ans. : Real Wages 100, 75, 62.5, 62, 60, 57.9, 56.8)

4. From the following data construct Real Wage Indices :

Year	Wages (Rs.)	Price Indices
1996	150	100
1997	170	150
1998	200	250
1999	250	300
2000	320	350
2001	360	380
2002	380	350
2003	400	360

Ans. : 100, 75.5, 53.3, 55.5, 60.9, 63.1, 72.4, 74.1

5. From the following data calculate Real Income.

Year	2001	2002	2003	2004	2005	2006	2007
Wage Indices	100	120	175	180	180	185	187.5
Price Indices	100	160	280	290	300	320	330

(Ans. Real Incomes : 100, 75, 62.5, 62, 60, 57.9, 56.8)

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Lesson 10

CONSUMER PRICE INDEX NUMBERS

10.0 OBJECTIVE

After studyhing this lesson you should be able to understand:

1. Consumer Price Index Numbers.

2. Methods of Construction of Index Numbers.

STRUCTURE OF LESSON

10.1 INTRODUCTION

The consumer price index numbers also known as cost of living index numbers are generally intended to represent the average change overtime in the prices paid by the ultimate consumer of a specified basket of goods and services. The need for constructing consumer price indices arises because the general index numbers fail to give an exact idea of the effect of the change in the general price level on the cost of living of different classes of people in different manners.

The consumer price index helps us to determining the effect of rise and fall in prices on different classes of consumers living in different areas. The construction of such an index is o great significance because very often the demand for a higher wage is based on the cost of living index and the wages and salaries in most of the countries are adjusted in accordance with the consumer price index.

The main object of cost of living index is to find out how much the consumers of a particular class have to pay more for a certain basket of goods and services in a given period compared to the base period. To bringout clearly this fact, the sixth International Conference of Labour Statisticians recommended that the term of cost of living index should be replaced in appropriate circumstances by the terms ‘price of living index’, cost of living price index, or ‘consumer price index’.

10.2 UTILITY OF THE CONSUMER PRICE INDICES

The consumer price indices are of great significance as can be seen from the following :

10.2.1 In the Regulation of Dearness Allowances and Bonus Policy: The different employers including the Government use this index number as an instrucment in fixing the dearness allowances and bonus payable to their employees from time time and avoiding the unpleasant situations from their employees thereby.

10.2.2 In the determination of Economic Policies : At government, level, the index numbers are used for wage policy, price policy, rent control, taxation and general economic policies.

10.2.3 In the determination of Purchasing Power : Cost of living Index number is used as a formidable instrument in the determination of purchasing power of money and value of real wages.

10.2.4 In the deflation of Income and Values : Cost of living index number is used, very often, in deflating the income or value series of a national account.

10.2.5 In the Analysis of Price Situation : Cost of living index number is also used in the analysis of price situations of a particular community.

10.2.6 In the Analysis of Markets : Index numbers are used for analysing markets for particular kinds of goods and services.

10.3 CONSTRUCTION OF CONSUMER PRICE INDEX

The following are the steps in constructing a Consumer Price Index :

10.3.1 Decision about the Class of People for whom the Index is meant : It is essential to decide clearly the class of people for whom the index is meant i.e., whether it relates to industrial wokers, teachers, officers, etc. The scope of the index must be clearly defined. For example, when we talk of teachers, we are referring to primary teachers, middle class teachers etc., or to all the teachers taken together. Along with the class of people it is also necessary to decide the geographical area covered by the index.

10.3.2 Conducting Family Budget Enquiry : The next step is to conduct a family budget enquiry covering the population group for whom the index is to be designed. The object of conducting a family budget enquiry is to determine the amount that an average family of the group included inthe index spends on diferent items of consumption. While conducting such an enquiry, therefore, the quantities of commodities consumed and their prices are taken into account. The consumption pattern can thus be easily ascertained.

10.3.3 Selection of Representative items : The next step is to select the representative items mostly consumed by that class of people selected. Such items are generally classified into the following five necessary groups :

i. Food

ii. Cloth

iii.Housing

iv. Fuel and Lighting

v. Miscellaneous Items

Further, each of these broad groups may again be detailed in certain sub-groups. For example the food group may be detailed under cereals (rice, wheat, etc.) pulses, meat, fish, milk and milk products, fats, oils, vegetables, sugar, spices, etc. The miscellaneous items generally include the important items like expenses on education, medicines, amusement, gifts and charities etc. The items of savings and investments are never included in such index numbers.

10.3.4 Collection of Price Quotations : The next step is to collect the prices of these commodities. For this the retail price quotations are to be obtained at regular weekly intervals from the ruling local markets viz fair price shops, super bazars etc.

10.3.5 Averaging the Price Quotations : The prices of commodities thus obtained from the markets should be averaged before they are taken to the construction of the index number.

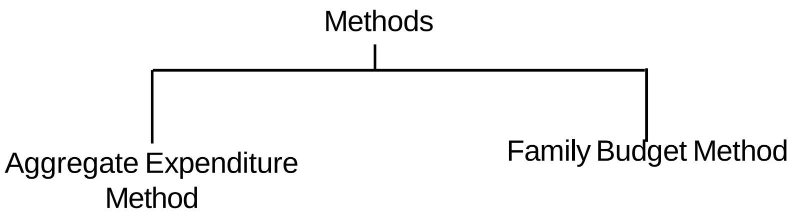
10.3.6 Assignment of Weights : Different weights are to be assigned rationally to the different items of consumption as it is done in case of any other index number.

10.4 METHODS OF CONSTRUCTING THE INDEX

The index may be constructed by applying any of the following methods.

1. Aggregate Expenditure Method or Aggregative method.

2. Family Budget Method or the Method of Weighted Relatives



10.4.1 Aggregate Expenditure Method : Under Aggregate Expenditure Method, the prices of commodities for various groups for the current year are multiplied by the quantities consumed in the base year and the aggregate expenditure incurrend in buying those commodities is obtained. In a similar manner the prices of the base year are multiplied by the quantities of the base year and aggregate expenditure for the base period is obtained. The aggregate expenditure of the current year is divided by the aggregate expenditure of the base year and the quotient is multiplied by 100. Symbolically

$$\text{Consumer Price Index} = \frac{\sum p_1 q_o}{\sum p_o q_o} \times 100$$

This is infact the Laspeyer method. This method is the most popular method for constructing consumer price index.

Illustration 1 :

From the following data construct consumer price index under aggregate expenditure method taking 2000 year as base year.

Commodity	Quantity in 2000	Unit (Rs.)	Price in 2000	Price in 2003 (Rs.)
Rice	6 Kgs	Kg	5.75	6.00
Wheat	6 Kgs	Kg	5.00	8.00
Green Grams	1 Kg	Kg	6.00	9.00
Oil	6 Kgs	Kg	8.00	10.00
Cloth	10 Mtrs	Mtr	2.00	1.50
Rent	1 house	One	20.00	15.00

Solution :

Commodity	Quantity in 2000	Unit	Price in 2000 (Rs.)	Price in 2003 (Rs.)	P_1q_o	P_oq_o
Rice	6 Kgs	Kg	5.75	6.00	36.00	34.50
Wheat	6 Kgs	Kg	5.00	8.00	48.00	30.00
Green Grams	1 Kg	Kg	6.00	9.00	9.00	6.00
Oil	6 Kgs	Kg	8.00	10.00	60.00	48.00
Cloth	10 Mtrs	Mtr	2.00	1.50	6.00	8.00
Rent	1 house	One	20.00	15.00	15.00	20.00
					174.00	146.50

$\sum P_1q_o =$ 174.00

$\sum P_oq_o =$ 146.50

$$\text{Consumer Price Index} = \frac{\sum p_1q_o}{\sum p_oq_o} \times 100$$

$$= \frac{174.00}{146.50} \times 100 = 118.77$$

Illustration - 2:

Construct consumer price index number from the following data:

Item	Prices in 2005	Value in 2005	Prices in 2006	Value in 2006
A	10	100	8	96
B	16	96	14	98
C	12	36	10	40
D	15	60	5	25

Solution :

Quantity is not given,

Hence Quantity = $\frac{\text{Value}}{\text{Price}}$

Item	Prices in 2005	Value in 2005	Prices in 2006	Value in 2006	P_1q_o	P_oq_o
A	10	100	8	96	80	100
B	16	96	14	98	84	96
C	12	36	10	40	30	36
D	15	60	5	25	20	60
					214	292

Consumer Price Index = $\frac{\sum p_1q_o}{\sum p_oq_o} \times 100$

$\sum p_1q_o = 214$, $\sum p_oq_o = 292$

$= \frac{214}{292} \times 100 = 73.29$

10.4.2 Family Budget Method : When this method is applied the family budgets of a large number of people for whom the index is meant are carefully studied and the aggregate expenditure of an average family on various items is estimated. These constitute the weights. The weights are thus the value weights obtained by multiplying the prices by quantities consumed (p_o x q_o). The price relatives for each commodity are obtained and these price relatives are multiplied by the value weights for each item and the product is divided by the sum of the weights. Symbolically

Consumer Price Index = $\frac{\sum PV}{\sum V}$

P = $\frac{P_1}{P_o}$ x 100 for each item

V = Value Weights i.e. P_o x q_o

Illustration - 3:

construct the consumer price index number for 2006 on the basis of 2005 from the following data using the family budget method.

Commodity	Quanity Consumed in 2005	Units	Price in 2005	Price in 2006
A	6 Quintal	Quintal	5.75	6.00
B	6 Quintal	Quintal	5.00	8.00
C	1 Quintal	Quintal	6.00	9.00
D	6 Quintal	Quintal	8.00	10.00
E	4Kg	Kg	2.00	1.50
F	1 Quintal	Quintal	20.00	15.00

Commodity	Quanity Consumed in 2005	Units	Prices in 2005	Prices in 2006	$\frac{P_1}{P_0} \times 100$	$P_o q_o$ V	PV
A	6 Quintal	Quintal	5.75	6.00	104.34	34.5	3600
B	6 Quintal	Quintal	5.00	8.00	160.00	30.0	4800
C	1 Quintal	Quintal	6.00	9.00	150.00	6.0	900
D	6 Quintal	Quintal	8.00	10.00	125.00	48.0	6000
E	4Kg	Kg	2.00	1.50	75.00	8.0	600
F	1 Quintal	Quintal	20.00	15.00	75.00	20.0	1500
						146.5	17400

$\sum V = 146.5$

$\sum PV = 17,400$

Consumer Price Index = $\frac{\sum PV}{\sum V} = \frac{17400}{146.5} = 118.17$

Illustration - 4

From the following Budget Analysis construct Consumer Price Index.

Item	Percentage of Expenditure	Prices in 2004	Price in 2005
Food	30	180	200
Rent	25	100	120
Cloth	15	70	90
Education	10	40	50
Others	20	70	100

Solution : Values = Percentage of Expenditure

Item	Percentage of Expenditure	Prices in 2004	Price in 2005 P_0	P $\{\frac{P_1}{P_0} x100\}$	PV
Food	30	180	200	111.11	3333.30
Rent	25	100	120	120.00	3000.00
Cloth	15	70	90	128.57	1928.55
Education	10	40	50	125.00	1250.00
Others	20	70	100	142.86	2857.20
	100				12369.05

$\sum V = 100$

$\sum PV = 12369.05$

$$\text{Consumer Price Index} = \frac{\sum PV}{\sum V}$$

$$= \frac{12369.05}{100} = 123.69$$

10.4.3 Index Number by using Geometric Mean : Under Family budget method, Index numbers can be computed by using geometric mean also. The Principal to compute Index number is

$$\text{Consumer Price Index} = \text{A. log of } \frac{\sum \log P.V.}{\sum V}$$

Illustration - 5 :

From the following data construct Consumer Price Index under family budget method by using Geometric Mean.

Item	Index numbers	Weight
Food	350	10
Cloth	150	2
Fuel	200	2
Rent	150	2
Others	225	4

Solution :

Item	Index numbers P	Weight V	log P	log P x V
Food	350	10	2.5441	25.4410
Cloth	150	2	2.1761	4.3522
Fuel	200	2	2.3010	4.6020
Rent	150	2	2.1761	4.3522
Others	225	4	2.3522	9.4088
		20		48.1562

$\sum V = 20$

$\sum \log P.V = 48.1562$

Consume Price Index = A. log of $\frac{\sum \log P.V}{\sum V}$

$= \text{A. log of} = \frac{48.1562}{20}$

= A. log of 2.40781

Consumer Price Index = 255.8

10.5 LIMITATIONS OF CONSUMER PRICE INDEX NUMBER

10.5.1 Consumer Price Index Number does not tell us anything about variations in living standard at two different places.

10.5.2 While constructing Index it is asumed that the basket does not change.

10.5.3 Like any other index the consumer price idex is based on a sample. It is often dificult to ensure perfect representativeness and in the absence of this the index may fail to provide the real picture.

10.6 LIMITATIONS OF INDEX NUMBER

Despite the importance of the index numbers in studying the economic and commercial activities and in measuring he relative changes in price level as the economic barometers, they suffer from certain limitations for which they should be very carefully used and interpreted. The following are some of the chief limitations among others.

10.6.1 Approximate Indicators : They are only approximate indicators of the change of a phenomenon viz., price level, quantity level, cost of living, production activity, etc., They never exactly represent the changes in the relative level of a phenomenon. This is because the index numbers are constructed mostly on the sample data.

10.6.2 Liable to be Misused : They are liable to be misused by a statistician with certain ulterior motive. For instance selection of wrong base year.

10.6.3 Errors at Every Stage : Index numbers are prone to embrance errors at each and every stage of construction viz.

1. Selection of the items
2. Obtaining the price quotation
3. Selection of the base year
4. Choice of the average
5. Assignment of Weights
6. Choice of the formula

10.6.4 Misrepresentation : They are liable to misrepresent the true picture of a phenomenon if the limited number of items chosen are not representative of the universe.

10.6.5 Improper Reflection : They are not capable of reflecting properly the relative changes in the quality level of the products, which very much change in modern times.

10.6.6 Purpose is Rigid : Index numbers are not capable of being used for any other purpose than the one for which they have been constructed particulaly. For instance a wholesale price index number can not measure the cost of living.

10.6.7 Bias : There is not such formula of Index number which is absolutely free from errors and limitations. For instance carpeyre's formula suffer from upward bias and Paasche's from downward bias.

10.7 EXERCISE

1. What is meant by consumer price index ? Explain its importance.
2. What are he steps in the construction of consumer price index.
3. Explain the methods of construction of consumer price index.

4. Describe the limitations of consumer price index.

5. From the following information calculate consumer price index.

Product	Quantity in 2004	Unit	Prices in 2004	Prices in 2005 (Rs. in Tens)
Paddy	20 Kgs	Kg	1.00	2.00
Wheat	50 Kgs	Kg	0.60	1.10
Oil	10 Kgs.	Kg	2.00	4.00
Ghee	500 gms	Kg	8.00	14.00
Sugar	5 Kgs.	Kg	1.00	1.80
Cloth	40 Mts.	Mtr	2.00	3.75
Rent	1 House	One	40.00	75.00

(Ans. : 1889-40)

6. Construct the Consumer's Price Index Number from the following data.

Group	Food	Fuel	Clothing	Housing	Other Items
Group Index for 2004	350	215	220	150	275
Expenditure %	46	7	10	12	25

(Ans. 274.63)

7. From the following data construct the index number for the food group.

Food	Rice	Wheat	Dal	Oil	Ghee
Indices	352	230	220	160	190
Weights	48	8	10	12	15

(Ans. : 276.41)

8. From the following data construct consumer price index number by using family budget method.

Item	Food	Fuel	Cloth	Rent	Others
Expenditure	35%	10%	20%	15%	20%
Prices in 2005 (Rs.)	150	125	75	30	40
Prices in 2006 (Rs.)	145	23	65	30	45

(Ans. : 97.87)

9. From the following data construct Consumer Price Index under Family Budget Method.

Item	A	B	C	D	E
Prices in 2004	10	20	35	50	100
Prices in 2005	25	32	70	65	120
Quantity in 2004	100	92	75	30	20

(Ans.: 178.31)

10. From the following data consruct consumer price index under family budget method by using geometric mean.

Item	Prices in 2004	Prices in 2005	Quantity in 2004
Wheat	2-00	2-50	40 Kgs
Sugar	3-00	3-25	20 Kgs
Milk	1-50	1-75	10 Ltrs.

(Ans. : 117.4)

11. What are he limitations of Index Numbers.

Dr. K. Kanaka Durga

Lesson 11

PROBABILITY

11.0 OBJECT

After completing the study of this lesson you are able to understand -

1. Meaning of Probability, and its importance
2. Types of events
3. Permutations and Combinations.
4. Addition Theorem and Multiplication Theorem.

STRUCTURE OF LESSON

11.1 Probability Origion and Meaning

Probability is an important branch of Mathematics that deal with the phenomenon of chance or random. In other words probability is a measure of uncertainty.

The theory of probability has its origin in gambling and games of chance. Its systematic study began in the seventeenth century, with the correspondence betwen two French mathematicians Blaise Pascal (1623 - 1662) and Pierre de Fermat (1601 - 1665). But the first attempt towards giving some mathematical rigor to the subject is credited to the noted French mathematician, astronomer and physicist Laplace (1749 - 1827).

The theory of probability, as we know it today, is credited to Andrei Nikolaevich Kolmogorov (1903 - 1987), a Russian analyst, topologist and probabilist. He laid the set theoretic foundation to probability in his fundamental work, 'Foundation of the Theory of Probability' in 1933.

11.2 Random Experiment

A random experiment is an experiment or operation. It produces some result or outcome. it deals with the following aspects :

1. The experiment can be repeated any number of times under identical conditions.
2. All possible outcomes of the experiment are known in advance, and
3. The actual outcome in a particular case is not known in advance.

Following are some of the examples of Experiment

a. Tossing a coin

b. rolling a die,

c. drawing a card from a pack of playing cards

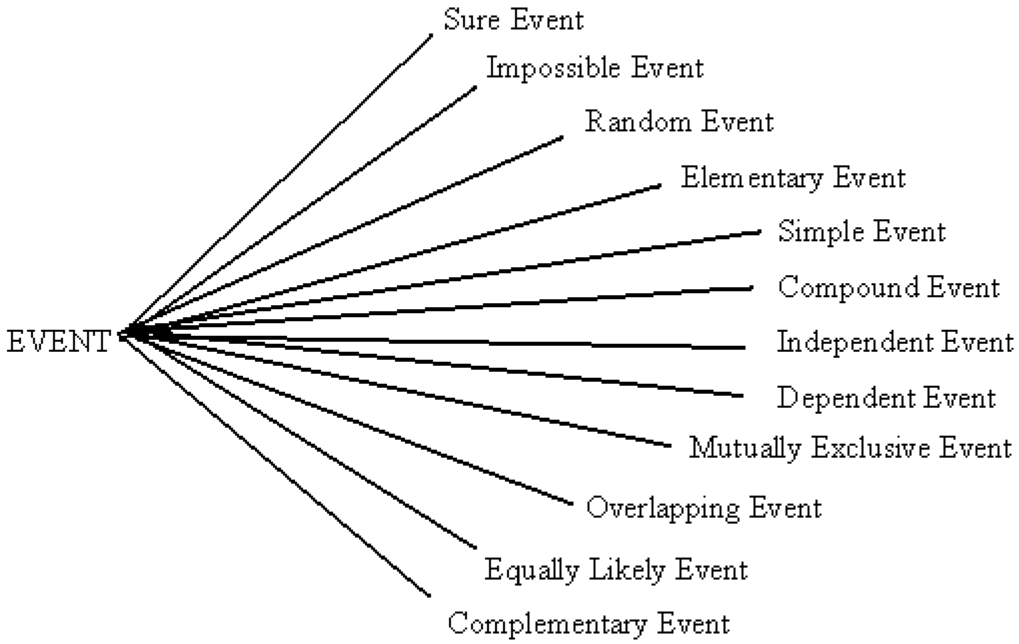
d. drawing a ball from a bag of different balls,

e. observing the defective items produced by a machine etc.,

11.3 Events - Types

An event is a case of interest which has the capability of happening, or taking place to a marked extent. it is a subset of a sample space for an experiemnt which is represented by some alphabet.

An event connected with the theory of probability may be divided into following types.



1 Sure Event : An event which is sure to take place when a random experiment is performed is called a sure, or a cetain event.

2 Impossible Event : An event which can never take place when an experiment is performed is called an impossible event.

3 Random Event : An event, the occurrence of which is uncertain or contingent is called a random or a contingent event.

4 Elementary Event : Every sample point is an elementary event.

5 Simple Event : A single event the probability of occurence of which is under consideration is called a simple event.

6 Compound Event : Two, or more events which occur jointly, and the probability of joint occurence of which is under consideration are called compound events.

7 Independent : An event the occurence of which does not depend upon the occurrence of any other event is called an independent event.

8 Dependent Event : A subsequent event, the probability, or occurrence of which is affected by the probability or occurrence of its preceding event or events is called a dependent event.

9 Mutually Exclusive Events : Two or more events are taken as mutually exclusive, if the happening of one excludes or prevents the happening of another at the same time. Example : with the tossing of a coin, either a head can come up or a tail can come up.

10. Overlapping Events : Two or more events that can take place together are called overlapping or not mutually exclusive events.

11. Equally likely Events : Two or more events are said to be equally likely, if each of them has the equal chance of happening.

12 . Complementary Events : Two events are said to be complementary to each other, if they are mutually exclusive and exhaustive as well.

Example : When a die is rolled, the outcome of an even number will be a complementary event to the outcome of an odd number.

Importance of Probability

Probability theory has been developed and employed to treat and solve many weighty problems. Following are the some uses of probability theory.

1 It is the foundation of the classical decision procedures of estimation and testing.

2 Probability models can be very useful for making predictions.

3 It is concerned with the construction of econometric models, with managerial decisions on planning and control.

4 It is involved in the observation of the life span of a radio active atom of the phenotypes of the off-springs, the crossing of two species of plants etc.,

5 It has become an indispensable tool for all types of formal studies that involve uncertainty.

6 The concept of probability is employed for various types of scientific investigations.

7 It is useful to solve many problems in everyday life.

In the words of ya-luu chou, probability is a method of decision making under uncertainty.

Rules of Probability

There are two important rules for computation of probabilities.

1. Rule of Multiplication : According to this rule, the probabilities of two or more related events are multiplied with each other to find out the net probability of their joint occurrence.

This rule is applied in the following areas.

a. The related events are independent of each other.

b. the related events are not mutually exclusive.

In such cases the net probability of the several related events say A, B and C is calculated as under.

P(AB) = P(A) X P(B)

P(ABC) = P(A) X P(B) X (P(C)

2. Rule of Summation : According to this rule, the probability of two, or more related events will be summed up to find out the probability of their joint occurrence.

This rule is applied in the following conditions.

a) The related events are dependent of each other.

b) The related events are mutually exclusive.

11.4 Axioms Of Probability

The Axiomatic approach to probability was introduced by the Russian Mathematician A.N.Kolmogorov in the year 1933. The whole field of probability theory for finite sample spaces is based upon the following three axioms.

1. The probability of an event ranges from zero to one. If the event cannot take place its probability shall be zero and if it is certain, i.e., bound to occur, its probability shall be one.
2. The probability of the entire sample space is 1 i.e. p(s) = 1.
3. If the A and B are mutually exclusive events then the probability of occurrence of either A or B denoted by $P(A \cup B)$ shall be given by

$P(A \cup B) = P(A) + P(B)$.

11.5 Classical definition of Probability

The Classical approach to probability is the oldest and simplest.

If a random experiment results in ‘n’ exhaustive, mutually exclusive and equally likely cases and ‘m’ of them are favourable to the happening of an event E then the probability of occurrance of E denoted by P(E) is defined by

$P(E) = \frac{m}{n}$

Since the number of results or outcomes not favourable to this event is n - m, the probability of non-occurrence of the event E, denoted by $P(E^c)$, is $\frac{n - m}{n}$

i.e. $P(E^c) = 1 - \frac{m}{n} = 1 - P(E)$

$\therefore P(E) + P(E^c) = 1$

It is clear from the definition that $0 \leq P(E) \leq 1$ for any event E.

11.6 Examples

Illustration 1

Ten dice are thrown, Find the probability that none of the dice shows the number 1.

Solution

We can express an outcome of this experiment as a list of 10 tuples formed from the symbols 1, 2, 3, 4, 5, 6. There are 6^{10} such entries and they are all equally likely.

Let 'A' be the event that none of the dice shows the number 1. The number of outcomes that donot have the number 1 is the number of 10 - tuples whose elements are chosen from the symbols 2, 3, 4, 5 and 6. The number of such 10 - Tuples is 5^{10}.

Therefore $P(A) = \frac{5^{10}}{6^{10}} = \left(\frac{5}{6}\right)^{10}$

Illustration 2

A number is picked from 1 to 20, both inclusive. Find the probability that it is a prime.

Solution

The total numbe of cases is 20. Let 'E' be the event that the number selected is prime. Then the numbe of favourable cases is equal to number of primes in the list 1, 2, 3, ... 20. The primes in this list are 2, 3, 5, 7, 11, 13, 17 and 19 which are 8 in number. Therefore the required probability is $\frac{8}{20}$ or $\frac{2}{5}$.

Illustration 3

A bag contains 4 red, 5 black and 6 blue balls. What is the probability that two balls drawn simultaneously are one red and one black ?

Total number of balls = 4 + 5 + 6 = 15.

Out of these 15 balls, 2 balls can be drawn in $15c_2$ ways

$\therefore$ Exhaustive numbe of cases is

$$15c_2 = \frac{14x15}{2} = 105$$

Out of 4 red balls 1 ball, can be drawn in $4c_1$ (= 4) ways and out of five black balls one ball can be drawn in $5c_1$ (=5) ways. Therefore the total number of favourable cases is 4 x 5 = 20.

$\therefore$ The required probability is $\frac{20}{105}$ or $\frac{4}{21}$.

11.7 PERMUTATIONS - COMBINATIONS

11.7.1 Permutations : Whenever there is importance to the arrangement or order in which the objects are placed it is a permutation. Permutation involves two steps called.

i) Selection

ii) Arrangement

For Example : Forming a three digit number using the digits 1, 2, 3, 4, 5 is a permutation. This involves two steps.

i) In the first step we select three digits, say 2, 4, 5.

ii) In the second step we arrange them to form a three digit number such as 245, 452 and 542 so on.

Factorial notation, will be useful in the calculation of the number of permutations. If n is a positive integer, definiton of n! or $\underline{n}$ by induction as follows

1! = 1 and n! = n (n - 1)! for n > 1

Example :

2! = 2(!) = 2 . 1 = 2

3! = 3(2!) = 3 . 2 = 6

4! = 4(3!) = 4 . 6 = 24

5! = 5(4!) = 5 . 24 = 120 etc.,

By convention, we define 0! = 1.

The number of permutations of n dissimilar things taken 'r' at a time is denoted by ^np_r and

$$^np_r = \frac{n!}{(n-r)!} \text{ for } 0 \leq r \leq n.$$

If n, r are positive integers and $r \leq n$, then i) $^np_r = n \cdot {}^{n-1}p_{r-1}$ (if $r \geq 1$)

ii) $^np_r = n \cdot (n-1) \cdot {}^{n-2}p_{r-2}$ (if $r \geq 2$)

11.7.2 Combinations : Whenever there is no importance to the arrangement or order but only selection is required. It is a combination. For instance forming a set with three elements using the digits 1, 2, 3, 4, 5 is a combination. This involves only one process, namely selection of three elements say, 2, 4, 5. Number of combinations is denoted by ^nc_r

$$= \frac{n!}{r!(n-r)!} = (n_r)$$

Example : $^5c_3 = \frac{5!}{3!.2!} = \frac{5x4}{2} = 10$

Illustration 4 :

Find the number of ways of selecting 7 members from a contingent of 10 soldiers.

Solution :

The number of ways of selecting 7 soldiers out of 10 is $^{10}c_7 = \frac{10!}{7!.3!} = 120$

Illustration 5 :

Find the number of ways of selecting 4 boys and 3 girls from a group of 7 boys and 6 girls.

Solution :

4 boys can be selected from the given 7 boys in $^7C_4$ ways and 3 girls from the given 6 girls in $^6c_3$ ways. Hence, by the fundamental principle, the number of ways of selecting 4 boys and 3 girls is

$^7C_4 \times {}^6C_3 = 35 \times 20 = 700$

Illustration 6 :

What is the probability of having 53 sundays in a leap year.

Solution :

Let us assume an event of 53 sundays in a leap year is 'A'.

Leap year consists of 366 days i.e. 52 weeks + 2 days

Those two days may be any one of the following.

Sunday, Monday

Monday, Tuesday

Tuesday, Wednesday

Wednesday, Thursday

Thursday, Friday

Friday, Saturday

Saturday, Sunday

No. of total events n = 7

No. of favourable events m = 2

$\therefore$ Probability P(A) = $\frac{2}{7}$

Illustration 7 :

From a pack of playing cards, one card is drawn at random. What is the pobability that it is either a Jack or an Ace.

Solution :

In a pack of playing cards, there are 52 cards in total, of which 4 are jacks and 4 are aces.

Thus, we have, $\sum n = 52$

n(J) = 4 and n(A) = 4

The probability that the card drawn is a Jack is given by

$$P(J) = \frac{n(J)}{\sum n} = \frac{4}{52} = \frac{1}{13}$$

Again, the probability that the card drawn is an Ace is given by

$$P(A) = \frac{n(A)}{\sum n} = \frac{4}{52} = \frac{1}{13}$$

The above two events being mutually exclusive, the probability of their compound happening is given by

$$P(J \text{ or } A) = P(J) + P(A) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Illustration 8 :

Mr X hits a target in 3 out of 4 shots and Mr. Y hits the target in 2 out of 3 shots. Find the probability of the target being hit at all when both of them try.

Solution :

Since Mr x hits the target in 3 out of 4, the probability that he fails to hit the same is given by

$q(X) = \frac{1}{4}$

Further, since Mr Y hits the target in 2 out of 3, the probability that he fails to hit the same is given by

q(Y) = $\frac{1}{3}$

Since the above two events are independent of each other, the probability of their compound non-happening is given by

q(X, Y) = q(X) . q(Y) = $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

We know, P + P = 1 so P = 1-q

Thus, the probability of the target being hit at all is given by

P(X, Y) = 1 - $\frac{1}{2} = \frac{11}{12}$

Illustration 9 :

Following is the data of wages of ,000 employees of XYZ Company.

Wages(Rs.)	No .of employees
1200 - 1400	9
1400 - 1600	118
1600 - 1800	478
1800 - 2000	200
2000 - 2200	142
2200 - 2400	35
2400 - 2600	18

From the above data select one person accidently what is the probability of earning wage in the following events.

1. Less than Rs, 1, 600

2. above Rs. 2,000

3. Between 1,600 and 2,000

Solution :

Selection of one from 1,000 employees = n = 1000

1. Event of earning wage less than, Rs. 1,600 = 'A'

No of employees in 'A' event = m = 9 + 118 = 127

Pobability , P(A) = $\frac{m}{n} = \frac{127}{1000}$

2. Event of earning wage above Rs. 2,000 = ‘B’

No. of employees in ‘B’ event m = 142 + 35 + 18 = 195

Probability, P(B) = $\frac{195}{1000}$

3. Event of earning wage between 1,600 and 2,000 is ‘C’

No. of employees in event ‘C’ = m = 478 + 200 = 678

Probability = P(C) = $\frac{678}{1000}$

Illustration 10 :

Following is the statement showing age and wages of 50 employees

Age (Years)	2500-3000	3000-3500	3500-4000	4000-4500	Total
20 - 30	8	3	-	-	11
30 - 40	2	5	2	2	11
40 - 50	-	2	9	6	17
50 - 60	-	-	6	5	11
Total	10	10	17	13	50

What is the probability of selecting an employee.

1. 30 - 40 years age group getting wage more than Rs.3500.

2. Below 40 years age group getting wage between 3000-3500.

Solution :

Selection of any one from 50 members is n = 50.

1. Event of selecting one person who is earning wage more than Rs, 3,500 and in the age group of 30-40 = E

Favourable methods to E = m = 2 + 2 = 4

Probability = P(E) = $\frac{4}{50}$ = 0.08

2. Event of selecting one person who is earning wage between 3000-3500 and in the age group of 30-40 = A.

Favourable methods to A = m = 3 + 5 = 8

Probability, P(A) = $\frac{m}{n} = \frac{8}{50}$ = 0.16

EXERCISE

1. What is Probability ? Explain its importance.

2. What are the types of events.

3. Define, permutations, and combinations.

4. A box contains 3 black and 5 white balls. If a ball is drawn at random, what is the probability that it is i)Black ii) White ?

$\left[(\text{i})\frac{3}{8},(\text{ii})\frac{5}{8}\right]$

5. A bag contains 5 white, 7 black and 4 red balls. If three balls are drawn at random, what is the probability that the three balls are white?

Ans. $\frac{5c_3}{16c_3}$

6. What is the probability of obtaining two tails and one head when three coins are tossed?

Ans. (3/8)

7. If 4 fair coins are tossed simultaneously, then find the probability that 2 heads and 2 tails appear.

Ans. (3/8)

8. If A, B are two independent events of a random experiment.

P(A and B) = P ($A \cap B$) = P(A) . P(B).

9. Find the Probability that a leap year contains 53 Sundays.

Ans. $\frac{2}{7}$

10. Find the probability that a non-leap year contains. i) 53 Sundays ii) 52 Sundays only

Ans. $\text{i})\frac{1}{7},\ \ \text{ii})\frac{6}{7}$

11. A page is opened arbitrarily from a book of 200 pages. What is the probability that the number of the page is a perfect square?

Ans. $\frac{7}{100}$

12. A bag contain 4 red, 5 black and 6 blue balls. What is the probability that two balls drawn simultaneously are one red and one black ?

Ans. $\frac{4}{21}$

13. If 3 English, 4 Telugu, 5 Hindi books are arranged in a shelf in one row, then find the probability that the books of the same language are side by side.

Ans. $\left(\frac{(3!)^2.4!.5!}{(12)!}\right)$

11.8 Addition Theorem on Probability

1. If E_1,E_2 are any two events of a random experiment then

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

2. If A, B, C are any events of a random experiment then

P A or B or C = $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) -$

$P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

3. If A, B are two disjoint events of a random experiment then

$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$

$\{\therefore\ A \cap B = \varphi \Rightarrow P(A \cap B)=0\}$

4. If A, B are two independent events of a random experiment.

$P(A \text{ and } B) = P(A \cap B) = P(A) . P(B)$

Illustration 1:

What is the Probability of throwing a total score of 7 with two dice?

Solution :

The sample space S of the experiment is given by

S = {(1,1), (1,2), ... , (1,6),

(2,1), (2,2), ... , (2,6),

(6,1), (6,2), ... , (6,6)}

In a typical element (2,3) $\sum$ S, first coordinate represents the score on first die, the second represents the score on the second die. S has 36 elemnts in it. All the points in S are equally likely.

Let E be the event of getting a total score 7.

The E = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)} and E has 6 elements.

$\therefore P(E) = \frac{6}{36} = \frac{1}{6}$.

Illustraion 2 :

What is the probability of obtaining two tails and one head when three coins are tossed?

Solution :

The sample space S of the experiment of tossing three coins is given by

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Let E be the event of obtaining two tails and a head then

E = {HTT, THT, TTH}

$\therefore P(E) = \frac{3}{8}$.

Illustration 3 :

A page is opened arbitrarily from a book of 200 pages. What is the probability that the number of the page is a perfect square?

Solution :

The sample space of the experiment in the question is given by

S = {1, 2, 3, ... 200}, So that n(S) = 200

Let E be the event of drawing a page with square number on it. Then

E = {1, 4, 9, ... 196} and n(E) = 14

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{14}{200} = \frac{7}{100}$

Illustration 4 :

If A and B are two events, then show that the Probability that exactly one of them occurs, is given by $P(A)+P(B)-2P(A\cap B)$

Solution :

P(exactly either of A, B occurs)

$= P((A - B) \cup (B - A))$

$= P((A \cap B^c) \cup (B \cap A^c))$

$= P(A \cap B^c) + P(B \cap A^c)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$(\therefore A = (A \cap B^C) \cup (A \cap B^C)$ and $(A \cap B) \cap A \cap B^C) = \varphi$

$= P(A) + P(B) - 2P(A \cap B)$

Illustration 5 :

What is the probability of drawing an ace or a spade from a well shuffled pack of 52 cards?

Solution :

A pack of cards means a pack containing 52 cards. 26 of them are red and 26 of them are black coloured. These 52 cards are divided into 4 sets, namely Hearts, Spades, Diamonds, and Clubs. Each set consists of 13 cards, namely A, 2, 3, 4, 5, 6, 7, 8, 9, 10, K, Q, J.

A = Ace, K = King, Q = Queen J = Jack

Let E_1 be the event of drawing a spade and E_2 the event of drawing an ace. Observe that E_1 and E_2 are not mutually exclusive. We have to find $P(E_1 \cup E_2)$. But $n(E_1) = 13$, $n(E_2) = 4$ and $n(E_1 \cap E_2) = 1$

$$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Illustration 6 :

From a pack of playing cards, one card is drawn at random. Find the probability that it is either a spade, or a club.

Solution :

There are 13 spade cards, 13 club cards and 52 total cards in a pack of playing cards. The probability of drawing a spade card is given by

$P(A) = \frac{13}{52} = \frac{1}{4}$ and

The probability of drawing a club card is given by $P(B) = \frac{13}{52} = \frac{1}{4}$

Thus, the probability of drawing either a spade, or a club card is given by

$$P(A \cup B) = P(A) + P(B)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Note that the two events related here are mutually exclusive events and so the principle of pure addition has been applied.

Illustration 7 :

A bag contains 4 green, 6 black and 7 white balls. One ball is drawn from that bg. Find the probability of having green ball or black ball.

Solution :

If the sample experiment of drawing one ball from the bag is

$n(S) = 17c_1 = 17$

Let E_1 be the event of drawing green ball

Let E_2 be the event of drawing black ball

$n(E_1) = 4c_1 = 4$, $n(E_2) = 6c_1 = 6$

$P(E_1) = \frac{4}{17}$, $P(E_2) = \frac{6}{17}$

E_1, E_2 are mutually disjoint events so $E_1 \cap E_2 = \varphi$

Probability of drawing green or black ball is

$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{4}{17} + \frac{6}{17} = \frac{10}{17}$ $\{ \therefore P(E_1 \cap E_2) = 0\}$

EXERCISE

1. A bag contains 4 white, 3 black and 5 red balls. If one ball is drawn, what is the probability of having white ball or black ball.

   (Ans. $\frac{7}{12}$)

2. From a pack of playing cards two cards are drawn at random one after the other without replacement. find the probability both of them are count cards.

   (Ans. $\frac{11}{51}$)

3. From a pack of playing cards one card is drawn at random. Find the probability that, the card is a King or a Queen.

   (Ans. $\frac{4}{52} + \frac{4}{52}$)

4. What is the probability of drawing an ace or a spade from a well shuffled pack of 52 cards?

   (Ans. $\frac{4}{13}$)

11.9 CONDITIONAL EVENT

If E_1, E_2 are two events of a random experiment. Then the event "happening of E_2 after E_1" is called a conditional event. It is denoted by $\frac{E_2}{E_1}$.

Similarly $\frac{E_2}{E_1}$ stands for the event happening of E_1 after the happening of E_2.

For instance : A business/man sells more Umbrellas during rainly season than otehr seasons. The probability of getting more profit depends on the condition that the season is rainy.

11.10 CONDITIONAL PROBABILITY

If E_1, E_2 are two events in a sample space then the event of happening E_2 after the evnt E_1 has occured is called Conditional Probability of E_2 given E_1. It is denoted by $P\left[\frac{E_2}{E_1}\right]$.

It is defined as $P\left[\frac{E_2}{E_1}\right] = P\left[\frac{E_1 \cap E_2}{P(E_1)}\right]$.

11.11 MULTIPLICATION THEOREM OF PROBABILITY (OR) THEOREM ON COMPOUND PROBABILITY

Let E_1, E_2 are two events in a sample space S such that, if

$P(E_1) \neq 0$, $P(E_2) \neq 0$, then

$P(E_1 \cap E_2) = P(E_1) . P\left[\frac{E_2}{E_1}\right]$

Illustration 1 :

A bag B_1 contains 4 white and 2 black balls. Bag B_2 contains 3 white and 4 black balls. A bag is drawn at random and a ball is chosen at random from it. Then what is the probability that the ball drawn is white?

Solution :

Let E_1, E_2 be the events of choosing bags B_1 and B_2 respectively. Then $P(E_1) = P(E_2) = \frac{1}{2}$

Let W be the event that the ball chosen from the bag selected is white.

Then $P(W/E_1) = \frac{4}{6} = \frac{2}{3}$. Similarly $P(W/E_2) = \frac{3}{7}$.

Observe that $W = (W \cap E_1) \cup (W \cap E_2)$ and $(W \cap E_1) \cap (W \cap E_2) = \varphi$

$\therefore$ $P(W) = P(W \cap E_1) + (W \cap E_2)$

$= P(E_1) . P(W/E_1) + P(E_2) . P(W/E_2)$

$$= \frac{1}{2}.\frac{2}{3}+\frac{1}{2}.\frac{3}{7}=\frac{1}{3}+\frac{3}{14}=\frac{17}{42}$$

Illustration 2 :

Suppose there are 12 boys and 4 girls in a class. If we choose three children one after another in succession. What is the probability that all the thre are boys?

Solution

Let E_1 be the event of choosing a boy child in i^{th} trail (i = 1, 2, 3) we have to find $P(E_1 \cap E_2 \cap E_3)$. By the multiplication theorem.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \,.\, P(E_2/E_1) \,.\, P(E_3/E_1 \cap E_2)$$

$$= \frac{12}{16} \text{x} \frac{11}{15} \text{x} \frac{10}{14} = \frac{11}{28}$$

Illustration 3 :

If one card is drawn at random from a pack of cards then show that getting an ace and getting heart are independent events.

Solution :

Let E_1 be the event of getting an ace and E_2 be the event of getting heart when a card is drawn from a pack of cards.

$$P(E_1) = \frac{4c_1}{52c_1} = \frac{4}{52} = \frac{1}{13},\ P(E_2) = \frac{13c_1}{52c_1} = \frac{13}{52} = \frac{1}{4}$$

$$P(E_1 \cap E_2) = \frac{1c_1}{52c_1} = \frac{1}{52} = P(E_1).\ P(E_2) \ldots$$

$\therefore$ A and B are independent.

Illustration 4 :

There are 3 black and 4 white balls in one bag, 4 black and 3 white balls balls in the second bag. A die is rolled and the first bag is selected if it is 1 or 3, and the second bag for the rest. Find the probability of drawing a black ball from the selected bag.

Solution :

Let E_1 be the event of selecting the first bag

and E_2 be the event of selecting the second bag,

$P(E_1)$ = Probability of getting 1 or 3 when a die is rolled = $2/6 = 1/3$.

$P(E_2) = 1 - \frac{1}{3} = \frac{2}{3}$

Now E_1, E_2 are mutually exclusive and exhaustive events.

Let E be the event of drawing a black ball from the selected bag.

Now $P(E/E_1)$ = Probability of selecting a black ball from the first bag = $3/7$.

$P(E/E_2)$ = Probability of selecting a black ball from the second bag = $4/7$.

$P(E) = P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) = \frac{1}{3} x \frac{3}{7} x \frac{2}{3} x \frac{4}{7} = \frac{11}{21}$

11.12 EXERCISE

1. What is probability that in rolling two dice the sum on the upper most faces exceeds 8, given that one of the faces is a 4?

   (Ans. $\frac{4}{11}$)

2. A pair of dice are rolled. What is the probability that neither die shows a 2 given that they sum to 7?

   (Ans. $\frac{2}{3}$)

3. A bag contains 7 red and 3 black balls. Two balls are drawn without replacement. Find the probability that the second ball is red if it is known that the first is red.

   (Ans. $\frac{2}{3}$)

4. The Probabilities of a problem being solved by three students are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Find the probability that the problem being solved.

   (Ans. $\frac{3}{4}$)

5. A bag contains 10 identical balls of which 4 are blue and 6 are red. Three balls are taken out at random from the bag one after the other. Find the probability that all the three balls drawn are red.

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LESSON - 12

BINOMIAL DISTRIBUTION

12.0 OBJECTIVE

After studying this lesson you should be able to understand.

1. Binomial Distribution its Meaning.
2. Characteristics and Importance.
3. Calculation of Binomial Distribution.
4. How it is useful in day-to-day events.

STRUCTURE OF LESSON

12.1 Frequency distribution - Definition, Types

12.2 Binomial Distribution - Definition

12.3 Binomial Distribution - Charcteristics

12.4 Importance of Binomial Distribution

12.5 ConstAnts of the Binomial Distribution

12.6 Illustrations

12.7 Exercise

12.1 FREQUENCY DISTRIBUTION

Frequency distribution can be defined as a summary presentation of a number of events of variables arranged according to their magnitudes. There are two types, they are

1. Frequency distributions on the values collected.
2. Frequency distributions of theories or theoretical distribution.

By theoretical distribution we mean a frequency distribution which is obtained in relation to a random variable by some mathematical models. They are

1. Binomial Distribution
2. Poisson Distribution
3. Normal Distribution

Theoretical Distribution

Binomial Distribution   Poisson Distribution   Normal Distribution

12.2 BINOMIAL DISTRIBUTION - MEANING

The Binomial Distribution also known as 'Bernoulli Distribution' is associated with the name of a Swiss Mathematician Jacob Bernoulli (1654 - 1705). Binomial Distribution is a probability distribution expressing the probability of one set of dichotomous alternatives i.e. success or failure.

'n' if one positive character $(q + p)^n = \sum_{x=0}^{n} nc_x \, q^{n-x} \cdot p^x$

$f(x) = nc_x (1-P)^{n-x} P^x$, x = 0, 12, 3 .....n

'n' if positive character O < P < 1 & f(x) >= 0

$$\sum f(x) = \sum_{x=0}^{n} {}^nc_x (1-P)^{n-x} P^x = 1$$

$P(x = x) = nc_x P^x q^{n-x} = nc_x q^{n-x} P^x$ ($\therefore$ p+q = 1)

$$\therefore \sum_{x=0}^{n} P(x = x) = 1$$

12.3 CHARACTERISTICS OF A BINOMIAL DISTRIBUTION

1. The shape and location of binomial distribution changes as P changes for a given 'n' or as 'n' changes for a given P. As P increases for a fixed n, the binomial distribution shifts to the right.
2. For 'n' trials, a binomial distribution consists of (n+1) terms, the successive binomial co-efficients being ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ${}^nC_3$..... ${}^nC_{n-1}$ and ${}^nC_n$
3. All the pobabilities of a binomial distribution can be obtained, if n and p are known, the value of q being 1-p.
4. If p = q = 1/2, the distribution obtained would be symmentrical and the expected frequencies on either side of the central would be identical. Even if $p \neq q$, the distribution would tend to be symmetrical.
5. A binomial distribution is positively skewed, if p<1/2 and it is negatively skewed if p>1/2.
6. A binomial distribution can be very well graphically represented by showing the values of the variable along the horizontal axis in terms of possible numbers of success, and the probability of occurences, i.e. the expected frequencies along the vertical axis.
7. A binomial distribution is unimodel if np is a whole number, i.e. an integer.
8. Since, the random variable X takes only integral values, a binomial distributions is a discrete probability distribution.

12.4 IMPORTANCE OF BINOMIAL DISTRIBUTION

1. The Binomial Distribution is a descrete probability distribution. But that is useful in describing an enoromous variety of real life events. For instance if an electrician want to know defective bulbs from a number of bulbs, he can know with the help of Binomial Distribution.
2. The Binomial Distribution has been used to describe a wide variety of processes in business and the social sceinces as well as other areas.
3. The Binomial Distribution cna be used when the outcome or resutls of each trial in the process are characterised as one of two types of possible outcomes.
4. It is also useful when the possibility of outcome of any trail does not change and is independent of the results of previous trails.
5. The probability of success remains constant from trail to trail.

12.5 CONSTANTS OF THE BINOMIAL DISTRIBUTION

The various constants of a binomial distributions that are built in some logical relationship may be listed as under.

1. Mean = np
2. Standard Deviation = $\sqrt{npq}$
3. Variance = npq
4. First moment about the mean, or $\mu_1 = 0$
5. Second moment about the mean or $\mu_2 = npq$
6. Third moment about the mean, or $\mu_3 = npq\,(q - p)$
7. Fourth moment about the mean or $\mu_4 = 2n^2p^2p^2 + npq(1 - 6pq)$
8. Moment Coefficient of Skewness :

$$\beta 1 = \frac{(q - p)^2}{npq}$$

And $$y_1 = \frac{q - p}{npq}$$

9. Coefficient of kurtosis :

$$\beta = 3 + \frac{1 - 6pq}{npq}$$

And $$y_2 = \frac{(1 - 6pq)}{npq}$$

Where

n = number of trails.

p = Probability of success or happening of an event

q = Probability of failure or non happening of an event.

12.6 ILLUSTRATIONS

Illustration 1 :

A coin is tossed six times, what is the probability of obtaining four or more heads?

Solution :

When a coin is tossed the probabilities of head and tail in case of an unbiased coin are equal i.e. p = q = 1/2

The various possibilities for all the events are the terms of the expansion $(q + p)^6$

$(q + p)^6 = q^6 + 6q^3$

The probability of obtaining 4 heads is

$15p^2q^4 = 15 \ (1/2)^2 \ (1/2)^4 = 0.234$

The probability of obtaining 5 heads is

$6qp^5 = 6 \ (1/2) \ (1/2)^5 = 0.095$

The probability of obtaining 6 heads is

$p^6 = (1/2)^6 = 0.016$

$\therefore$ The probability of obtaining 4 or more heads is

0.234 x 0.095 x 0.016 = 0.345

Illustration 2:

A coin is tossed 5 times what is the probability of obtaining 2 heads and 3 tails.

Solution :

If probability of head = p

If probability of tail = q

p = 1/2, q = 1/2

When a coin is tossed 5 times the probability of getting 2 heads and 3 tails

$= 5C_2 \ (1/2)^3 \ (1/2)^2 = 5/16$

Illustration 3 :

The mean of a binomial distribution is 50 and standard deviation is 5. Calculate n, p and q.

Solution :

Th mean of binomial distribution = np = 50

Standard Deviation of binomial distribution = $\sqrt{npq} = 5$

Since np = 50, npq = 25

50q = 25

q = 25/50 = 1/2 = 0.5

p = 1 - q

i.e. 1 - 0.5

i.e. 0.5

np = 50

n (0.5) = 50

n = 50/0.5

= 100

Hence the solution n = 100, p = 0.5, q = 0.5

Illustration 4:

Twelve unbiased coins are tossed simultaneously. Find the pobability of obtaining exactly 8 heads.

Solution :

If p denotes the probability of a head, then p = q = 1/2. Here n = 12, if the random variable X, the probability of 8 heads is given by

$P(r) = p(x = r) = {}^{n}C_r\, p^r\, q^{n-r}$

$= {}^{12}C_r\, (1/2)^r\, (1/2)^{12-r}$

$= {}^{12}C_r\, (1/2)^{12}$

$= 1/4096\ {}^{12}C_r$

Illustration 5:

Three balanced coins are tossed simultaneously. Find the chances of getting 3,2,1 and 0 heads.

Solution :

By the model of binomial distribution. We have,

$(p + q)n = pn + {}^{n}C_1\, p^{n-1}\, q^1 +$

${}^{n}C_2\, p^{n-2}\, q^2 + {}^{n}C_3\, p^{n-3}\, q^3$

Where p = probability of getting a head = 1/2

q = probability of not getting a head = 1/2

and n = number of trails at a time = 3 coins substituting the respective values in the above model we get

$(1/2 + 1/2)^3 = (1/2)^3 + {}^{3}C_1\, (1/2)^2 \,.\, (1/2)^1 + {}^{3}C_2\, (1/2)^1 \,.\, (1/2)^2 + {}^{3}C_3\, (1/2)^o \,.\, (1/2)^3$

$= 1/8 + 3(1/2)^3 + 3(1/2)^3 + (1/2)^3$

$= 1/8 + 3/8 + 3/8 + 1/8$

Hence, the required chances of getting 3,2,1 and 0 heads are 1/8, 3/8, 3/8 and 1/8 respectively.

Illustration 6:

If on an average, rail falls on 12 days in every 30 days, find the probability:

i) that the first 4 days of a given week will be fine, and the remainder wet.

ii) That rain will fall on just 3 days of a given week.

Solution:

i) The probability of a rainy day, or p = 12/30 or 2/5 Thus q = 1 - 2/5 = 3/5

ii) The probability that the first four days will be find = $(3/5)^4$

The probability that the next 3 days will be wet = $(2/5)^3$

The above two probabilities being independent of each other, their compound probability is given by

$(p)^4 \times (p)^3 = (3/5)^4 \times (2/5)^3 = 81/625 \times 8/125 = 648/78125 = 0.008$

The probability that rain will fall on just 3 days of a given week is given by

$p(r) = {}^nC_r\, p^r\, q^{n-r}$

Where n = number of days in a week i.e. 7

r = number of rainy days expected i.e. 3

p = 2/5, and q = 3/5

Thus $p(3) = {}^7C_3\, (2/5)^3\, (3/5)^4$

= 35 x 8/125 x 81/625 = 22680/78125 = 0.29

Hence, the probability that rain will fall in just 3 days of a given week = 0.29.

Illustration 7:

If the probability of a deecive article is 1/10, find the Mean, variance, moment co-efficient of skewners for the binomial distribution of the defective articles in a total of 40.

Solution:

(i) Mean X = np

Where n = 40, p = 1/10

X = 40 x 1/10 = 4

ii) Variance

This is given by = X = npq

Where p = 1/10, thus q = 1 - 1/10 = 9/10 and n = 40

V = 40 x 1/10 x 9/10 = 3.6

Illustration 8:

A candidate arives at the following answers to a problem involving a binomial distribution.

X = 2.4, V = 3.2 comment on the data.

Solution :

For a binomial distribution,

Mean is given by np i.e. 2.4

and Variance is given by npq i.e. 3.2

Thus q = npq / np = 3.2 / 2.4 = 1.33

But according to the theory of probability.

Since (p + q) = 1, 1 can never be > 1 i.e. 1.33

Hence, the data relating to the distribution are not consistent.

12.7 EXERCISE

1. What is Binomial Distribution.

2. What are the characteristics of Biomial Distribution.

3. Explain the importance of Binomial Distribution.

4. Eight coins are thrown simultaneously. Show that the probability of obtaining at least 6 heads is 37/ 256.

(Ans. : $\frac{37}{256}$ is proved)

5. If the chance of suffering from the occupational hazards is 25%. What is the probability, that out of every 6 workers, 4 or more will suffer from the hazards.

(Ans. : 0.038)

6. In Binomial Distribution, Mean = 20. Variance = 15 findout Q.

(Ans. : 0.25)

7. One Binomial Distribution Mean = 6 and Standard Deviation is $\sqrt{2}$. Prove that Binomial Distribution is $(1/3 + 2/3)^9$.

8. 7 Coins are tossed what is the probability to get (i) Exactly 4 heads, (ii) More than 4 heads.

9. In a Binomial Distribution n = 15, p = ∩, 1, p (x>=2). Find out

(Ans. $1-15C_o\, q^{15} p^o + 15C_1\, q^{14} p^1$)

10. Prove that Probability is 5/ 32 if 5 coins are tossed at a time.

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LESSON 13

POISSON DISTRIBUTION

13.0 OBJECTIVE

After studying this lesson you should be able to understand.

1. What is Poisson Distribution.

2. Features and its usage.

STRUCTURE OF LESSON

13.1 POISSON DISTRIBUTION - MEANING

Poisson Distribution is a discrete probability distribution and is very widely used in statistical work. It was orginated by a French mathematician, Simeon Denis Poisson (1781-1840) in 1837. Poisson Distribution may be expected in cases where the chance of any individual event being a success is small.

The Poisson Distribution can be a reasonable approximation of the binomial under certain conditions like :

(i) number of trials i.e. n is indefinitely large, i.e., $n \to \propto$

(ii) P i.e., the probability of success for each trial is indefinitely small, i.e. $p \to 0$

(iii) np = m(say) is finite.

Definition :

The Poisson Distribution is defined as $p(r) = \frac{e^{-m}.m^{r}}{r!}$

Where, e =2.7183, a constant i.e. the base of the natural logarithm.

m = mean of the distribution i.e. np

And r = expected number of success viz : 0, 1, 2, 3, ...n

But for finding the successive terms of 0, 1, 2, 3, ...n events the following expansion model is to be applied.

$$\sum p(r) = e^{-m}[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + ...]$$

13.2 Features of Poisson Distribution

The essential features of Poisson Distribution may be enumerated as under.

1) It is a discrete probability distribution, since it is concerned with the occurrences that can take only integral values like 0,1, 2, ... $\propto$.

2) It has a single parameter m (Mean) with which only all the possible probabilities of the Poisson Distribution can be easily obtained.

3) It is a reasonable approximation of the binomial distribution where, $n \to \propto$, $p \to 0$ and m is finite.

13.3 ASSUMPTIONS

Following are the assumptions of theory of Poisson Distribution.

1. The happening or non-happening of any event does not effect the happening, or non-happening of any other event.

2. The probability of happening of more than one event in a very small interval is negligible.

3. The probability of a success for a short time interval or for a short space is preportional to the length of the time interval, or space is proportional to the length of the time interval, or space interval as the case may be.

13.4 IMPORTANCE OF POISSON DISTRIBUTION

The Poisson Distribution is used in practice in a wide variety of problems where ther are infrequently occurring events with respect to time, area, volume of similar units. Following are the areas where Poisson Distribution is useful.

(i) It is used in quality control statistics to count the number of defects of an item.

(ii) It is used in biology to count the number of bacteria.

(iii) It is used in physics to count the number of particles emitted from a radio active substance.

(iv) It is used in Insurance sector to count the number of casualities.

(v) It is used in waiting-time problems to count the number of incoming telephone calls or incoming customers.

(vi) It is used to know number of traffic arrivals such as trucks at terminals, aeroplane at airports, ships at docks, etc.,

(vii) In problems dealing with the inspection of manufactured products with the probability that any one piece is defective is very small and the lots are very large.

(viii) It is used to model the distribution of the number of persons joining a queue to receive a service or purchase of a product.

Constants of the Poisson Distribution

The various useful constants of the Poisson Distribution that have been built up on mathematical relationship are as under :

1. Mean. or $\overline{X} = m$

2. Variance = m

3. Standard Deviation or $\sigma = \sqrt{m}$

4. $\mu_1 = 0$

5. $\mu_2 = m$

6. $\mu_3 = m$

7. $\mu_4 = m + 3m^2$

8. $\beta_1 = \frac{\mu_3^{\,2}}{\mu_2^{\,3}} = \frac{m^2}{m^3} = \frac{1}{m}$

9. $\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{m + 3m^2}{m^2} = 3 + \frac{1}{m}$

10. $y_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{m}}$

11. $y_2 = \beta 2 - 3 = 3 + \frac{1}{m} - 3 = \frac{1}{m}$

13.5 ILLUSTRATIONS

Illustration 1:

In a district on an average one house in 1,000 has a fire during a year. If there are 2,000 houses in that district, what is the probability that exactly 5 houses will have a fire during the year.

Solution :

Applying the Poisson Distribution

$\overline{X}$ = np

n = 2000, p = $\frac{1}{1000}$

np = 2000 x $\frac{1}{1000}$ = 2

$$p(r) = e^{-m}.\frac{m^r}{r!}$$

Here m = 2, r = 5 and e = 2.7183

$$p(5) = 2.7183^{-2} x \frac{2^5}{5!}$$

$$= \frac{Rec.[anti\log(2x\log 2.7183)]\,x\,32}{5x4x3x2x1}$$

$$= \frac{Rec.[anti\log(2x.4343)]\,x\,32}{120}$$

$$= \frac{Rec.[anti\log(.8686)]\,x\,32}{120}$$

$$= \frac{Rec.[7.389]\,x\,32}{120} \quad \frac{0.1352x\,32}{120} = .036$$

Illustration 2:

A factory produces blades in packets of 10. The probability of a balde to be defective is 0.3%. Find the number of packets having three defective blades in a consignment of 10,00 packets.

Solution :

m = np = 10 x 0.3%

= 10 x 0.003 = 0.03

p (x = 3 defects) = $\frac{e^{-0.03}.(0.03)^3.}{3!}$

$$= \frac{0.9704x0.000027}{3x2x1} = 0.0000044$$

Therefore, the total number of packets having three defective baldes in a consignment of 10,000 packets = 10000 x 0.0000044 = 0.044.

Illustration 3:

There are an average of 4 accidents taking place per month on busy read. Find the probability that there will be less than 4 accidents in a year.

Solution :

No. of average accidents m = 4

Random variable denotes the number of accidents on road per month.

$$p = (x = r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-4}.4^r}{r!}$$

The required probability there will be less than 4 accidents is given by

p(x<4) = p(x = 0) + p(x = 1) + p(x = 2) + p(x = 3)

= p(x = 2) + p (x = 3)

$$= e^{-4}\left[1 + 4x\frac{42}{21} + \frac{43}{3!}\right]$$

$$= e^{-4}\left[1 + 4 + 8 + 10.67\right]$$

= $e^{-4}x23.67$ = 0.0183 x 23.67 = 0.4332

Illustration 4:

If a random number. X follows the Poisson pattern such that p(x = 1) = p(x = 2),

find a) The Mean of the distribution

b) p(x = 0)

Solution :

a) Mean of the Distribution

By the Poisson model of probability we have, $p(r) = e^{-m}.\frac{.m^r}{r!}$

Thus p(x = 1) = $e^{-m}.\frac{.m}{1!} = e^{-m}.m$

And p(x = 2) = $e^{-m}.\frac{m^2}{2!} = e^{-m}.\frac{m^2}{2}$

According to the data p(x = 1) = p(x = 2)

$$e^{-m}.m = e^{-m}.\frac{m^{-2}}{2}$$

$1 = \frac{m}{2}$ $\therefore$ m = 2

Hence, Mean of the distribution = 2

b) p(x = 0)

By the poisson's model of probability we have

$$p(r) = e^{-m}.\frac{m^{-r}}{r!}$$

$$p(0) = e^{-2}.\frac{2^0}{0!}$$

$p(0) = e^{-2}.\frac{2^0}{0!}$ = e^{-2} x 1 = 0.1353 x 1 = 0.1353

The above value of e^{-2} has been obtained by using a calculator. If we look to the e^{-m} Table with reference to 2, the value will be 0.1353 for e^{-2}.

Illustration 5:

Between the hours of 2 and 4 p.m. the average number of phone calls per minute coming into the switch board of a company is 2.5. Find the probability that during a particular minute, there will be no phone call at all. (Given e^{-2} = 0.13534, and e^{-5} = 0.60650).

Solution :

Given, the average number of phone calls, or m = 2.5

e^{-2} = 0.13534, and e^{-5} = 0.60650

$\therefore$ $e^{-2.5}$ = e^{-2}x e^{-5} = 0.13534 x 0.60650 = 0.08208

According to the Poisson's model of probabilities we have, $p(r) = e^{-m}.\frac{m^r}{r!}$

where, r = number of events expected i.e. 0,

e^{-m} = 0.08208, Calculated above.

And m = value of the mean i.e. 2.5.

Thus p(0) = 0.08208 x $\frac{2.5^0}{0!}$ = 0.08208

Illustration 6:

If a random number x follows the Poisson pattern such that

p (x = 2) = 9p(x = 4) + 90p(x = 6)

Find out value of λ and x mean.

Solution :

p(x = x) = $\frac{e^{-\lambda}.\lambda^x}{x!}$, x= 0, 1, 2, ... λ > 0

Here p(x = 2) = 9p(x = 4) + 90p (x = 6)

$$\frac{e^{-\lambda}.\lambda^2}{2!} = e^{-n}\left[9\frac{\lambda^4}{4!}+90\frac{\lambda^6}{4!}\right]$$

$$= \frac{e^{-\lambda}.\lambda^2}{8}[3\lambda^2 + \lambda^4]$$

$$= \lambda^4 + 3\lambda^2 - 4=0$$

Therefore $\lambda^2 = \frac{-3\pm\sqrt{9+16}}{2} = \frac{-3\pm 5}{2}$

x >0 , So $\lambda^2 = 1 \Rightarrow \lambda = 1$

M = λ = 1 and μ = Variance = λ = 1

Illustraiton 7:

The following mistakes per page were observed in a book:

No. of mistakes per page	0	1	2	3	4
No. of times the mistake occured	211	90	19	5	0

Solution:

Filling Poisson Distribution.

	f	fx
0	211	0
1	90	90
2	19	38
3	5	15
4	0	0
	N = 325	$\sum fx$ = 143

$$X = \frac{\sum fx}{N} = \frac{143}{325} = 0.44$$

Mean of the distribution or m = 0.44

$(P_0) = e^{-m} = 2.183^{-.44}$

Rec. [anti log (.44 log 2.7183)]

Rec. [anti log (0.44 x 0.4343)]

Rec.[anti log .1910] = Rec. 1.552 = 0.6443

$N\ P_0 = 0.443 \times 325 = 209.40$

$$N(P_1) = N(P_0) \times \frac{m}{1} = 209.4 \times 0.44 = 92.14$$

$$N(P_2) = N(P_1) \times \frac{m}{2} = 92.14 \times \frac{0.44}{2} = 92.14 \times 0.22 = 20.27$$

$$N(P_3) = N(P_2) \times \frac{m}{2} = 0.27 \times \frac{0.44}{3} = 6.76 \times 0.44 = 2.97$$

$$N(P_4) = N(P_3) \times \frac{m}{4} = 2.97 \times \frac{0.44}{4} = 2.97 \times 0.11 = 0.33$$

The expected frequencies of Poisson Distribution are :

x	0	1	2	3	4	
f	209.40	92.14	20.27	2.97	0.33	= 325.4

Illustration 8:

Suppose that a manufactured product has 2 defects per unit of product inspected. Using Poisson distribution, calculate the probabilities of finding a product without any defect, 3 defects and 4 defects.

(Given $e^{-2} = 0.135$)

Solution :

Average number of defects or m = 2.

$P(r) = \frac{e^{-m}\, m^r}{r!}$

$P(0) = e^{-2} = 0.135$ given

$P(1) = (P_0) \times m = 0.135 \times 2 = .27$

$P(2) = (P_1) \times \frac{m}{2} = 0.27 \times \frac{2}{2} = 0.27$

$P(3) = P(_2) \times \frac{m}{3} = 0.2 \times \frac{2}{3} = 0.18$

$P(4) = P(_3) \times \frac{m}{4} = 0.18 \times .5 = 0.09$

13.6 EXERCISE

1. What is Poisson Distribution.
2. Explain features of Poisson Distribution.
3. What are the assumption of Poisson Distribution.
4. Describe the importance of Poisson Distribution.
5. Find the probabilities, and the expected frequencies for the following data according to the "Law of improbable events given by Poisson.

No. of Mistakes	:	0	1	2	3	4
No. of Printed pages	:	109	65	22	3	1

Ans.: Mean = 0.61, Total of the expected frequencies = 200

6. Compute the probabilities and fit a Poisson Distribution for the following data.

No. of Accidents	:	0	1	2	3	4
No. of Days	:	40	35	15	6	4

Ans.: Mean = 0.99, Total of the expected frequencies = 100

7. Between the hours 10 AM and 12 noon the average number of phone calls per minute coming into the switch board of a company is 2.45. Find the probability that during one particular minute there will be at most 3 phone calls.

   Ans. : 0.76799

8. It is given that 3% of electric bulbs manufactured by a company are defective. Using the Poisson approximation. Find the probability that a sample of 100 bulbs will contain :

   i) no defective, ii) exactly one defective

Ans. : 0.15

9. Certain mass-produced articles of which 0.5 percent are defective are packed in cartons each containing 130 articles. What proportion of cartons are free from defective articles and what proportion contain 2 or more defectives? Given $e^{-0.5} = 0.6068$

Ans. : 0.09025

[Therefore the proportion of cartons free from defective articles is 60.65% or 61% and with 2 or more defectives is 9%(approx.).]

Dr. K. Kanaka Durga

Lesson 14

NORMAL DISTRIBUTION

14.0 OBJECTIVE

After studying this lesson you should be able to understand..

1. What is Normal Distribution.

2. How it is useful.

STRUCTURE OF LESSON

14.1 NORMAL DISTRIBUTION

Normal Distribution which is also known as Normal Probability Distribution was first discovered by 'De Moivre' (1667 - 1754) as the limiting form of the binomial model in 1773. Later it was discovered and used extensively in both natural and social sciences by the French mathematician 'Laplace' (1749-1827). Subsequently, the theory was highly appreciated and used by Karl Friedrich Gauss (1777 - 1855) in the calculation of orbits of heavenly bodies. On his name this theory is also known as 'Gaussian' theory.

14.2 NORMAL DISTRIBUTION - MEANING

The Normal Distribution is most useful theoretical distribution for continuous variables. Many statistical data concerning business and economic problems are displayed in the form of normal distribution. It is corner stone of modern statistics. It is measured by the following equation.

$$P(x) = \frac{1}{\sigma\sqrt{2\Pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Where μ = Mean of the normal random variable X.

σ = Standard variation of the given normal distribution.

$\sqrt{2\Pi}$ = 2.5066

e = 2.7183

From this mathematical equation it must be seen that Mean (μ), and Standard Deviation (σ) ar ethe only two parameters of the Normal Distribution.

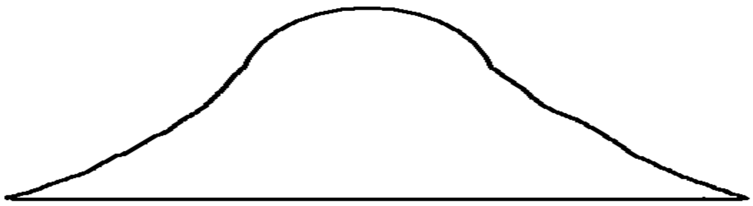
14.3 PROPERTIES/ FEATURES / CHARACTERISTICS

1. The normal distribution can have different shapes depending on different values of μ and σ but there is one and only one normal distribution for any given pair of values for μ and σ. Usuallay the normal curve is bell-shaped and symmetrical in its appearance.

2. Normal distribution is a limiting case of Poisson Distribution when it mean 'm' is large.

3. The mean of a normally distributed population lies at the centre of its normal curve.

4. The two tails of the normal distribution never touch the horizontal axis, as shown in the following figure.



5. Normal distribı

i) n $\to \propto$ ε

ii) neither p nor q is very small.

6. Since there is only one maximum point, the normal curve is unimodel i.e. it has only one mode.

7. The variable distributed according to the normal curve is a continuous one.

8. The first and the third quarties (Q_1, Q_3) are equidistant from the median.

14.4 SIGNIFICANCE OF NORMAL DISTRIBUTION

Significance of Normal Distirbution is clear from the following points.

1. The normal distribution has numerous mathematical properties which make it popular and comparatively easy to manipulate.
2. The nomal distribution is used extensively in statistical quality control in industry in setting up of control limits.
3. It is useful in large sampling theoy to find out the estimates of the parameters from statistics and confidence limits.

4. This distribution can be regarded as a limiting case of the binomial distribution if the value of 'n' (numbe of trials) is very large, and neither p nor q is very small.

5. In the words of W.J.Youden, the importance of normal distribution is described artistically in the shape of a normal curve as under.

Normal

Law of Eorrors

Experience of Mankind

As one of the Broadest

Generalisation of Natural

Philosophy. It serves as the

Guiding instrument in Researches

In the Physical and Social Sceinces and

In Medicine, Agriculture and Engineering.

It is an indispensable tool for the analysis and interpretation

Of the Basic Date obtained by observation and experiment

14.5 Constants of Normal Distribution

The various constants that related to Normal Distribution are listed below.

1. Mean represented by μ.
2. Standard Deviation represented by σ.
3. Variance represented by V or σ^2.
4. $\mu_1 = 0$.
5. $\mu_2 = \sigma^2$.
6. $\mu_3 = \mu_4 = 3\sigma^4$.
7. $\beta = \frac{\mu_3^2}{\mu_2^3} = 0$.
8. $\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3$.
9. $y = \sqrt{\beta_1} = \frac{\mu_3}{\sqrt{\mu_2^3}} = 0$

10. Z (a random normal variate) = $\frac{X-\mu}{\sigma}$

X = The value of a random variable

μ = The Mean

σ = Standard Deviation

14.6 Illustrations

Illustration 1

Indicate the area in a normal curve when z = 1.75

Solution

With reference to the Area table of the standard normal curve, we find that the area enclosed between the μ and z at 1.75 = 0.4599. This is exhibited through the shaded portion of the following figure.

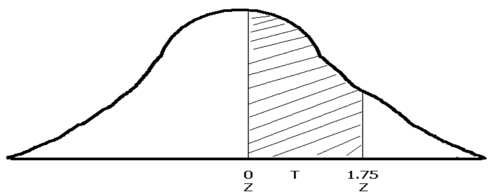


Illustration 2

Locate the area in a normal curve when z = -1.96.

Solution

With reference to the Area table of the normal curve, we find that the area enclosed between the μ and z at -1.96 = 0.4750. This is indicated through the shaded portion of the following figure.

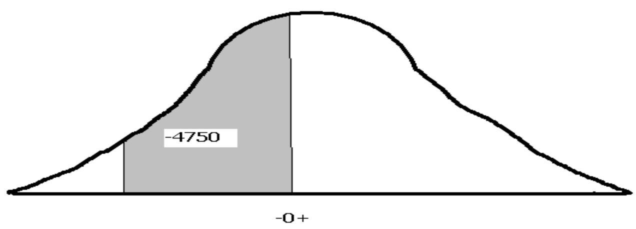


Illustration 3

Show the area in a normal curve which is more than z = 1.5.

Solution

We know that in a normal curve, the total area to the right of z at 0 = 0.5 and with refernce to the area table of the normal curve, we find that the area between z at 0 and z at 1.5 = 0.4332.

Thus by equation, the area to the right of 2 at 1.5 (i.e. more than z = 1.5).

0.5 - 0.4332 = 0.0668

This is indicated through the shaded portion in the diagram drawn as under:

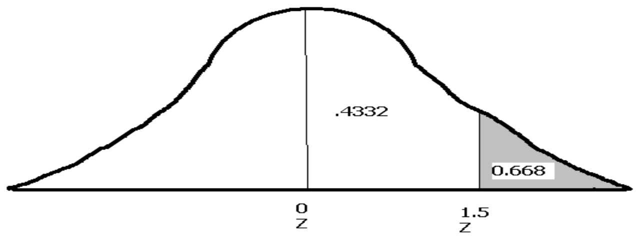


Illustration 4

Find the area in a normal curve to the left of z at -1.60.

Solution

We know that in a normal curve, the total area to the left of z at 0 = 0.5 and with reference to the Area table of the Normal curve we find that the area between z at 0 and z at -1.6 = 0.4452.

Thus by equation, the area to the left of z at -1.6 = 0.5 - 0.4452 = 0.548. This is indicated through the shaded portion in the following figure.

The positive and negative signs of z do not affect the area as indicated in the Area table of a normal curve in as much as the curve is symmentrical.

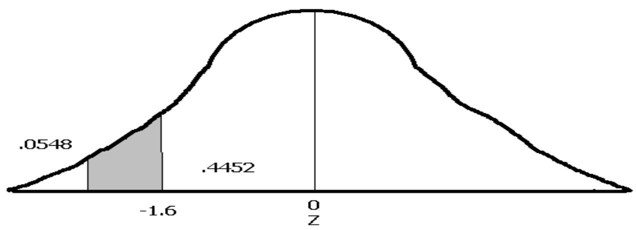


Illustration 5

If X, N(2,25) find out P(0<x<10), (-8<x, 1)

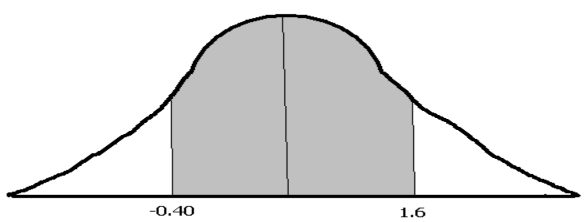
Solution

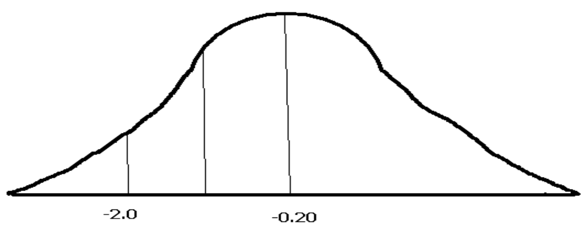
$$P(0<x<10) = P\left[\frac{-2}{5} < 2 < \frac{8}{5}\right]$$

Because, z = $\frac{x-2}{5}$

$$\int^{1.6} \varphi(z)dz = \int_{0}^{1.6} \varphi(z)dz + \int_{0}^{0.4} \varphi(z)dz$$

= 0.4452 + 0.1554 = 0.6006 (From the table)





Like that P(-8 < x < 1) = P(-z < z < -0.2)

$$\int_{-\infty}^{-0.2} \varphi(z)dz = \int_{-\infty}^{-2.0} \varphi(z)dz$$

= 0.4772 - 0.0793 = 0.3979 (From tables)

Illustration 6

Assume the mean height of soldiers to be 68.22 inches with a variance of 10.8 inches. How many soldiers in a regiment of 1,000 would you expect to be over six feet tall?

Solution

Assume that the distribution of height is normal.

Standard normal variate or z = $\frac{x - \bar{x}}{\sigma}$

X = 72 iches

$\bar{X}$ = 68.22

$\sigma = \sqrt{10.8} = 3.286$

z = $\frac{72 - 68.22}{3.286}$ = 1.15

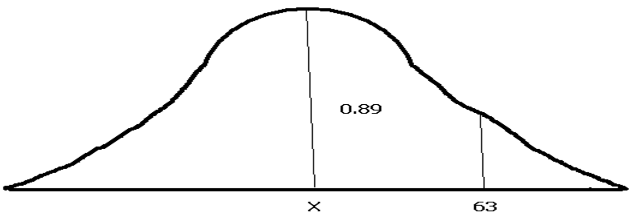
Area to the right of the ordinate at 1.15 from the normal table is (0.5000 - 0.3749) - 0.1251. Hence the probability of getting soldiers above six feet is 0.1251 and out 1000 solidiers the expectation is 0.1251 x 1000 or 125.1 or 125. Thus the expected number of soldiers over six feet tall = 125.

Illustration 7

In a distribution exactly normal 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution?

Solution

Since 7% of items are under 35, 43% are between $\overline{X}$ and 35. Similarly the percentage of items between $\overline{x}$ and 63 is 39%.



. The standard normal variate corresponding to 0.43(43%) is 1.448.

Thus $\frac{35-\overline{X}}{\sigma}$ = -1.48

The standard normal variate corresponding, to 0.39 (39%) is 1.23.

$\frac{63-\overline{X}}{\sigma}$ = + 1.23

From (i) and (ii) 1.48 σ - $\overline{X}$ = -35

1.23 σ - $\overline{X}$ = 63

On additing these equations, we get 2.71 σ = 28

$\therefore\ \sigma = \frac{28}{2.71}$ = 10.33

1.48 x 10.33 - $\overline{X}$ = -35

- $\overline{X}$ = -35 - 15.3

$\overline{X}$ = 50.3

Hence the mean of the distribution is 50.3 and σ = 10.33.

Illustration 8

The income of a group of 10,000 persons was found to be normally distributed with mean = Rs.750 p.m. and standard deviation = Rs.50. Show that of this group about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs. 823. What was the lowest income among the richest 100?

Solution

Standard Normal Variate or z = $\frac{x - \overline{x}}{\sigma}$

Hence, X = 668, $\overline{X}$ = 750, σ = 50

Z = $\frac{668 - 750}{50}$ = $\frac{-82}{50}$ = -1.64

Area to the right of the ordinate at - 1.64 is (0.4495 + 0.5000) = 0.9495

$\therefore$ The expected number of persons getting above Rs. 668.

= 10,000 x 0.94950 = 9495.

This is about 95% of the total i.e. 10,000.

The standard normal variate corresponding to 832 is $Z = \frac{832 - 750}{50}$ = $\frac{82}{50}$ = 1.64

Area to the right of ordinate at 1.64 is

0.5000 - 0.4495 = 0.0505

The number of persons getting above Rs. 832 is

10000 x 0.0505 = 505

This is approximately 5%.

Probability of getting richest 100.

= $\frac{100}{10000}$ = 0.01

Standard normal variate having 0.01 area to its right = 2.33

X 33 = $\frac{X - 750}{50}$

X = (2.33 x 50) + 750 = Rs.866.5

Illustration 9

1,000 light bulbs with a mean life of 120 days are installed in a new factory their length of life is normally distributed with standard deviation 20 days.

1. How many bulbs will expire in less than 90 days.

2. If it is decided to replace all the bulbs together, what interval should be allowed between replacemnts if not more than 10% should expire before replacement ?

Solution

1. $\overline{X}$ = 120 σ 20 x = 90

Standard Normal Variate or Z = $\frac{90 - 120}{20}$ = -1.5

Area of the curve at (Z = -1.5) up to the mean ordinate = 0.4332

Area of the left of -1.5 - 0.5 - 0.4332 = 0.0668

Number of bulbs expected to expire in less than 90 days

= 0.0668 x 1000 = 66.8 or 67

2) The value of standard normal variate corresponding to an area 0.4 (0.5 - 0.1) is 1.28.

$\frac{X - 120}{20}$ = -1.28

X = 120 - (1.28 x 20)

= 120 - 25.6 = 94.4 or 94

Hence the bulbs will have to be replaced after 94 days.

Illustration 10

The Bombay Municipal Corporation installed 2,000 bulbs in the streets of Bombay. If these bulbs have an average life of 1,000 burning hours, with a Standard Deviation of 200 hours, what number of bulbs might be expected to fail in the first 700 burning hours? The table of area of the normal curve at selected values is as follows:

$\frac{X - \overline{X}}{\sigma}$	Probability
1.00	0.159

1.25	0.106
1.50	0.067

Solution

Average life of bulbs $\overline{X}$ = 1000 hours, σ = 200 hours, X = 700 burning hours

$$Z = \frac{X - \overline{X}}{\sigma} = \frac{700 - 1000}{200} = -1.5$$

Area to the left of (-1.5) = 0.067

$\therefore$ Number of bulbs expected to fail in the first 700 hours = 0.067 x 2000 = 134.

14.8 Exercise

1. What is meant by Normal Distribution. Explain its importance.
2. Explain the properties of Normal Distribution.
3. What are the constants of Normal Distribution.
4. Explain the features of Area of Normal Distribution.
5. If X, N (2,25). Find out P(0<x<10), (-8<x<1)

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Lesson - 15

CALCULAS - DIFFERENTIATION - I

OBJECTIVES:

After studying this chapter, you will be able to understand.

- the derivative and write the derivatives of standard functions.
- differentiate functions using standard derivatives and rules of differentiation.
- higher order derivatives of function.
- derivatives as a rate measure.

STRUCTURE:

15.1 Introduction

15.2 Differentiation - Derivatives

15.3 Derivative of composite function

15.4 Exercises.

15.1 INTRODUCTION:

The word calculus stands for the method of computation. There may be an arithmetic calculus or a probability calculus. The most common use of calculus is in regard to the computation of the rate of change in one variable with reference to an infinitesimal variation in the other variable. For example we know that given the speed, the distance covered is a function of time or given the distance, the time taken is a function of speed. Then there is a dependent variable which gets an impulse of change by a change in the independent variable. Calculus gives us the technique for measuring these changes in the dependent variable with reference to a very small change, approaching almost zero, in the independent variable or variables. The techniques concerning the calculation of the average rate of change are studied under differentiation or the Differential calculus and the calculation of the total amount of change in the given range of values is studied under integration or integral calculus.

The usefulness of both these is very great in business. Given certain functional relations, we can find out the average rate of change in the dependent variable with reference to a change in one or more independent variables. For example with a given demand function, it would be possible to find the degree of change in demand with reference to a small change in price or incene or both as the case may be and also the maximum and minimum values of the function.

15.2 DIFFERENTIATION - DERIVATIVES :

1. Differentiation :

To express the rate of change in any function we have the concept of derivative which involves infinitesimally small changes in the dependent variable with reference to a small change in independent variable.

Differentiation we can say is the process of finding out the derivative of a continuous function. A derivative is the limit of the ratio of the increment in the function corresponding to a small increment in the argument as the latter tends to zero.

Let y = f(x) be a continuous function. The rate of change in y is δy when the rate of change in x is δx.

$$\therefore \delta y = f(x+\delta x)$$

$$\therefore y+\delta y = f(x+\delta x)$$

$$\delta y = f(x+\delta x) - y$$

$$\delta y = f(x+\delta x) - f(x)$$

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$\underset{\delta x \to 0}{Lt} \frac{\delta y}{\delta x}$ is denoted by $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$ $f^1(x)$ when h = δx

$\frac{dy}{dx}$ means the rate of change in y w.r.t 'x'

If y = f(x) be a function then $\frac{dy}{dx} = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$

Derivative of x^n :

$$\because f(x) = x^n$$

$$\because f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \overset{Lt}{h \to 0} \ \frac{(x+h)^n - x^n}{h}$$

$$= \overset{Lt}{h \to 0} \ \frac{(x^n + {}^n c_1 x^{n-1}.h + {}^n c_2 x^{n-2}.h^2 + ... + h^n) - x^n}{h}$$

$$= \overset{Lt}{h \to 0} \ ({}^n c_1 x^{n-1} + {}^n c_2 x^{n-2} h + ... + h^{n-1})$$

$= n\ x^{n-1}$

$\frac{d}{dx}$ **(x^n) = n. x^{n+1}**

Derivative of e^x :

$let\ f(x) = e^n$

$$\therefore\ f^1(x) = \overset{Lt}{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \overset{Lt}{h \to 0} \ \frac{e^{x+h} - e^x}{h}$$

$$= \overset{Lt}{h \to 0} \ e^x \frac{(e^h - 1)}{h}$$

= e^x **(1)**

= e^x $\left(\because \overset{Lt}{h \to 0} \ \frac{(e^h - 1)}{h} = 1 \right)$

$\frac{d}{dx}$ **(e^x) = e^x**

Derivative of const 'c' :

$Let\ f(x) = c$

$$\therefore\ f^1(x) = \overset{Lt}{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \overset{Lt}{h \to 0} \ \frac{(c) - c}{h}$$

$= 0$

$\frac{d}{dx}$ **(c) = 0**

Derivative of $\sqrt{x}$:

$$let\ f(x) = \sqrt{x}$$

$$\because\ f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \underset{h \to 0}{Lt} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \underset{h \to 0}{Lt} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{x}{\sqrt{x+o} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Derivative of log x :

$$let\ f(x) = \log x$$

$$\because\ f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\log\ (x+h) - \log x}{h}$$

$$= \underset{h \to 0}{Lt} \frac{1}{h} \log \left(\frac{x+h}{x}\right)$$

$$= \underset{h \to 0}{Lt} \log \left(\frac{x+h}{x}\right)^{\frac{1}{h}}$$

$$= \frac{1}{x} . \underset{h \to 0}{Lt} \log \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} \log \underset{h \to 0}{Lt} \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} \log e$$ log x means $\log_e x$

$$= \frac{1}{x} (1)$$

$$= \frac{1}{x}$$

$\frac{d}{dx}$ **(log x)** $= \frac{1}{x}$

Derivative of Sin x:

$$let\ f(x) = Sinx$$

$$\therefore f^1(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sin(x+h) - \sin x}{h}$$

$$= \underset{h \to 0}{Lt} \frac{2 \operatorname{Cos}\left(\frac{(x+h)+x}{2}\right) \operatorname{Sin}\left(\frac{(x+h)-x}{2}\right)}{h}$$

$$= \underset{h \to 0}{Lt}\ 2 \operatorname{Cos}\left(x + \frac{h}{2}\right) \frac{\operatorname{Sin}\frac{h}{2}}{\frac{h}{2}.2}$$

= Cos (x + 0) . 1

= Cos x

$\frac{d}{dx}$ **(Sin x)** = Cos x

Derivative of UV :

$$let\ f(x) = u(x).\ v(x)$$

$$\therefore\ f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{u(x+h)\ v(x+h) - u(x).\ v(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{u(x+h)\ v(x+h) - u(x+h)\ u(x) + v(x+h)\ v(x) - u(x).\ v(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{u(x+h)\left[v(x+h) - v(x)\right] + v(x)\left[u(x+h) - u(x)\right]}{h}$$

$$= \underset{h\to 0}{Lt}\ u(x+h).\ \underset{h\to 0}{Lt} \frac{v(x+h) - v(x)}{h} \underset{h\to 0}{Lt}\ v(x).\ \underset{h\to 0}{Lt} \frac{u(x+h) - u(x)}{h}$$

$$= u(x).v^1(x) - v(x)\ u^1(x)$$

$$\therefore \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Derivative of $\frac{u}{v}$:

$$\text{let } f(x) = \frac{u(x)}{v(x)}$$

$$\therefore\ f^1(x) = \underset{h\to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h}$$

$$= \underset{h\to 0}{Lt} \frac{u(x+h)\ v(x) - v(x+h).\ u(x)}{h\ v(x+h).\ v(x)}$$

$$= \underset{h\to 0}{Lt} \frac{u(x+h)\ v(x) - u(x)\ v(x) + u(x)\ v(x) - v(x+h).\ u(x)}{h\ v(x+h).\ v(x)}$$

$$= \underset{h\to 0}{Lt} \frac{v(x)\left[u(x+h) - u(x)\right] - u(x)\left[v(x+h) - v(x)\right]}{h\ v(x+h)\ v(x)}$$

$$= \left[v(x).\underset{h\to 0}{Lt} \left(\frac{u(x+h) - u(x)}{h} \right) - u(x).\left(\underset{h\to 0}{Lt} \frac{v(x+h) - v(x)}{h} \right) \right].$$

$$\underset{h\to 0}{Lt} \frac{1}{v(x+h) \cdot v(x)}$$

$$= \frac{v(x).\ u^1(x) - u(x)\ v^1(x)}{[v(x)]^2}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

Formula :

1. $\frac{d}{dx}(x^n) = n.\ x^{n-1}$

2. $\frac{d}{dx}(e^x) = e^x$

3. $\frac{d}{dx}(a^x) = a^x \cdot \log a$; $\frac{d}{dx}(ax) = a$

4. $\frac{d}{dx}(\log x) = \frac{1}{x}$

5. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$

6. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$, $\frac{d}{dx}\frac{1}{x^n} = \frac{-n}{x^{n+1}}$

7. $\frac{d}{dx}(c) = 0$

8. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

9. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

10. $\frac{d}{dx}$ (uvw) = uv$\frac{dv}{dx}$w+ uw$\frac{du}{dx}$(v) + vw$\frac{du}{dx}$(u)

11. $\frac{d}{dx}$ **(**Sin x **)** = Cos x

12. $\frac{d}{dx}$ **(**Cosx **)** = -Sin x

13. $\frac{d}{dx}$ **(**tanx **)** = Sec^2 x

14. $\frac{d}{dx}$ **(**Cot x **)** = - $Cosec^2$ x

15. $\frac{d}{dx}$ **(**Sec x **)** = Sec x. tan x

16. $\frac{d}{dx}$ **(**Cosec x **)** = - Cosec x. Cot x

SOLVED EXAMPLES :

i) Find the derivative of x^5

let y = x^5

$\frac{dy}{dx} = \frac{d}{dx}(x^5)$

$= 5\ x^{5-1}$

$=5x^u$

ii) Find the derivative of $x^{2/3} + 2x^3 + 7x + 5$

let y = $x^{2/3} + 2x^3 + 7x + 5$

$\frac{dy}{dx} = \frac{d}{dx}(\ x^{2/3} + 2x^3 + 7x + 5\ )$

$= \frac{d}{dx}(\ x^{2/3}) + 2\frac{d}{dx}(\ x^3) + \frac{d}{dx}(7x) + \frac{d}{dx}(\ 5\ )$

$= {}^2/_3\ x^{2/3\ -1} + 2.3\ x^{3-1} + 7 + 0$

$= {}^2/_3\ x^{-1/3} + 6x^2 + 7$

iii) Find the derivative of $\frac{1}{x^2}$

let $y = \frac{1}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= \frac{-2}{x^{2+1}}$$

$$= \frac{-2}{x^3}$$

iv) Find the derivative of x^3 + Sin x + e^x

let y = x^3 + Sin x + e^x

$$\frac{dy}{dx} = \frac{d}{dx}\ (x^3 + \text{Sin } x + e^x)$$

$$= \frac{d}{dx}(x^3) + \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)$$

= $3x^2$ + Cos x + e^x

v) Find the derivative of $\frac{x^3 + 2x + 7}{x^2}$

let $y = \frac{x^3 + 2x + 7}{x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\ \left(\frac{x^3 + 2x + 7}{x^2} \right)$$

$$= \frac{d}{dx}\left(x + \frac{2}{x} + \frac{7}{x^2}\right)$$

$$= \frac{d}{dx}(x) + 2\ \frac{d}{dx}\left(\frac{1}{x}\right) + 7\ \frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$= 1+2\left(\frac{-1}{x^2}\right)+\left(\frac{-2}{x^{2+1}}\right)$

$1-\frac{2}{x^2}-\frac{2}{x^3}$

vi) Find the derivative of $\left(x-\frac{1}{x}\right)^3$

Let y = $\left(x-\frac{1}{x}\right)^3$

$\frac{dy}{dx}=\frac{d}{dx}\left(x-\frac{1}{x}\right)^3$

$= \frac{d}{dx}\left(x^3-\frac{1}{x^3}-3.x^2.\frac{1}{x}+3x.\frac{1}{x^2}\right)$

$= \frac{d}{dx}\left(x^3-\frac{1}{x^3}-3x+\frac{3}{x}\right)$

$= \frac{d}{dx}(x^3)-\frac{d}{dx}\left(\frac{1}{x^3}\right) - 3\frac{d}{dx}(x) + 3\frac{d}{dx}\left(\frac{1}{x}\right)$

$= 3x^2 -\left(\frac{-3}{x^4}\right) - 3(1) + 3\left(\frac{-1}{x^2}\right)$

$= 3x^2 -\frac{3}{x^4} - 3 - \frac{3}{x^2}$

vii) Find the derivative of x^2. Sinx

Let y = x^2. Sinx

$\frac{dy}{dx}=\frac{d}{dx}(x^2.\text{Sin}x)$

$= x^2\frac{d}{dx}(\text{Sin}x) + \text{Sin}\,x\frac{d}{dx}(x^2)$

$= x^2 \text{Cos}\, x + \text{Sin}x\, .2x$

$= x^2 \text{Cos}\, x + 2x \text{ Sin } x$

vii) Find the derivative of $(x^2 + x + 1) \log x$

Let $y = (x^2 + x + 1) \log x$

$$\frac{dy}{dx} = \frac{d}{dx} [(x^2 + x + 1) \log x]$$

$$= (x^2 + x + 1) \frac{d}{dx} (\log x) + \log x \frac{d}{dx} (x^2 + x + 1)$$

$$= (x^2 + x + 1) \left(\frac{1}{x}\right) + \log x (2x + 1 + 0)$$

$$= (x + 1 + \frac{1}{x}) + (2x + 1) \log x$$

viii) Find the derivative of $[(x^2 -3) (4x^3+1)]$

Let $y = (x^2 -3) (4x^3+1)$

$$\frac{dy}{dx} = \frac{d}{dx} [(x^3 -3) (4x^3+1)]$$

$$= (x^3 -3) \frac{d}{dx} (4x^3+1) + (4x^3+1) \frac{d}{dx} (x^3 -3)$$

$$= (x^3 -3) [4.\, 3x^2 + 0] + [4x^3 + 1] (3x^2 + 0)$$

ix) Find the derivative of $[(\sqrt{x} -3x) (x + \frac{1}{x})]$

Let $y = [(\sqrt{x} -3x) (x + \frac{1}{x})]$

$$\frac{dy}{dx} = \frac{d}{dx} [(\sqrt{x} -3x) (x + \frac{1}{x})]$$

$$= (\sqrt{x} -3x) \frac{d}{dx} (x + \frac{1}{x}) + (x + \frac{1}{x}) \frac{d}{dx} (\sqrt{x} -3x)$$

$= (\sqrt{x}\ \ -3x)\ (x + \frac{-1}{x^2}) + (x + \frac{1}{x})\ [\frac{1}{2\sqrt{x}} - 3\ (1)]$

$= (\sqrt{x}\ \ -3x)\ (x - \frac{1}{x^2}) + (x + \frac{1}{x})\ [\frac{1}{2\sqrt{x}} - 3]$

x) Find the derivative of e^x log x – Sin x

Let y = e^x log x – Sin x

$\frac{dy}{dx} = \frac{d}{dx}$ e^x log x – Sin x

$= e^x \frac{d}{dx}$ (log x) + log x $\frac{d}{dx} e^x$ – Cos x

$= e^x . \left(\frac{1}{x}\right)$ **+** log x . e^x – Cos x

$= \frac{e^x}{x}$ + e^x . log x – Cos x

xi) Find the derivative of x e^x Sin x

Let y = x e^x Sin x

$\frac{dy}{dx} = \frac{d}{dx}$ (x e^x Sin x)

$= x\, e^x \frac{d}{dx}$ (Sin x) + x Sin x $\frac{d}{dx}$(e^x) + e^x Sin x $\frac{d}{dx}$(x)

= x e^x Cos x **+** x .e^x . Sin + e^x Sin x

xii) Find the derivative of $\frac{x^2+2}{x^2-x-2}$ with respected to ' x '

Sol : $\frac{d}{dx}\left[\frac{x^2+2}{x^2-x-2}\right]$

$$= \frac{\left(x^2-x-2\right)\frac{d}{dx}\left(x^2+2\right)-\left(x^2+2\right)\frac{d}{dx}\left(x^2-x-2\right)}{\left(x^2-x-2\right)^2}$$

$$= \frac{\left(x^2-x-2\right)(2x)-\left(x^2+2\right)(2x-1)}{\left(x^2-x-2\right)^2}$$

$$= \frac{2x^3-2x^2-4x-2x^3+x^2-4x+2}{\left(x^2-x-2\right)^2}$$

$$= \frac{-x^2-8x+2}{\left(x^2-x-2\right)^2}$$

xiii) Find the derivative of $\frac{a-b\ Cos\ x}{a+b\ Cos\ x}$ with respected to ' x '

Sol : $\frac{d}{dx}\left[\frac{a-b\ Cos\ x}{a+b\ Cos\ x}\right]$

$$= \frac{\left(a+b\ Cos\ x\right)\frac{d}{dx}\left(a-b\ Cos\ x\right)-\left(a-b\ Cos\ x\right)\frac{d}{dx}\left(a+b\ Cos\ x\right)}{\left(a+b\ Cos\ x\right)^2}$$

$$= \frac{\left(a+b\ Cos\ x\right)\left(b\ \text{Sin}\ x\right)-\left(a-b\ Cos\ x\right)\left(-b\ \text{Sin}\ x\right)}{\left(a+b\ Cos\ x\right)^2}$$

$$= \frac{ab\ Sin\ x+b^2\ \text{Sin}\ x\ Cos\ x+ab\ Sin\ x-b^2\ \text{Sin}\ x\ Cos\ x}{\left(a+b\ Cos\ x\right)^2}$$

$$= \frac{2ab\ Sin\ x\ Cos\ x}{\left(a+b\ Cos\ x\right)^2}$$

xiv) Find the derivative of $\frac{x^2}{\text{Sin } x} - \frac{a^x}{2x-3}$ with respected to ' x '

Sol : $\frac{d}{dx}\left[\frac{x^2}{\text{Sin } x} - \frac{a^x}{2x-3}\right]$

$$= \frac{d}{dx}\left[\frac{x^2}{\text{Sin } x}\right] - \frac{d}{dx}\left[\frac{a^x}{2x-3}\right]$$

$$= \frac{Sin\ x\ \frac{d}{dx}\left(x^2\right) - x^2 \frac{d}{dx}(\text{Sin } x)}{(\text{Sin } x)^2} - \frac{(2x-3)\ \frac{d}{dx}\left(\text{a}^x\right) - a^x \frac{d}{dx}(2x-3)}{(2x-3)^2}$$

$$= \left[\frac{Sin\ x\ .\ 2x - x^2 Cos\ x}{\text{Sin}^2\ x}\right] - \left[\frac{(2x-3)\ a^x \log a - a^x.2}{(2x-3)^2}\right]$$

$$= \frac{2x\ Sin\ x\ \text{-}\ x^2 Cos\ x}{\text{Sin}^2\ x} - \frac{(2x-3)\ a^x \log a - 2a^x}{(2x-3)^2}$$

Exercise :

Find the derivative of the following with respect to ' x '

1. $3x^2+5x -1$
2. 2 Cos x - 4 e^x + 5 log x
3. a^x + log x + $\sqrt{x}$
4. x^7- 7 e^x + tan x
5. $\sqrt{2}$ Sin x + 7 x^6 + $\frac{3}{x^2}$
6. log a^x
7. x^3 log x
8. (x^2 + 1) Cot x
9. $4x^3$ - 3 Sin x log x + e^x log x
10. a^x tan x log x

11. $\dfrac{x(x+2)}{2x+5}$

12. $\dfrac{Cos\ x}{Sin\ x+Cos\ x}$

13. $\dfrac{Cos\ x}{x^2} - \dfrac{e^x}{5x+4}$

15.3 DERIVATIVE OF COMPOSITE FUNCTION :

If ' f' is a differentiable function at 'x' and g is differentiable function at x then gof is differentiable at x and $(gof)^1 (x) = g^1 [f(x)]. f^1(x)$.

Solved examples :

1) Find the derivative of $(5x - 6)^3$ with respected to ' x '

Sol : let $y = (5x - 6)^3$

$$\frac{dy}{dx} = \frac{d}{dx}(5x - 6)^3$$

$$= 3. (5x - 6)^{2}. \frac{d}{dx}(5x - 6)$$

$$= 3. (5x - 6)^{2}. 5$$

$$= 15. (5x - 6)^2$$

2) Find the derivative of a^{2x+3} with respected to ' x '

Sol : let $y = a^{2x+3}$

$$\frac{dy}{dx} = \frac{d}{dx}(a^{2x+3})$$

$$= a^{2x+3} . \log a. \frac{d}{dx}(2x + 3)$$

$$= a^{2x+3} \log a. 2$$

$$= 2. a^{2x+3} \log a.$$

3) Find the derivative of Cosec^4 x with respected to ' x '

Sol : let y = Cosec^4 x

$$\frac{dy}{dx} = \frac{d}{dx}(\text{Cosec}^4 x)$$

$$= \frac{d}{dx}(\text{Cosec } x)^4$$

$$= 4 \text{ Cosec}^3 x \, . \, \frac{d}{dx} (\text{Cosec } x)$$

$$= 4 \text{ Cosec}^3 x \, . \, (- \text{Cosec } x. \text{ Cot } x)$$

$$= - 4 \text{ Cosec}^4 x \, . \, \text{Cot } x$$

4) Find the derivative of $20^{\log \tan x}$ with respected to ' x '

Sol : let y = $20^{\log \tan x}$

$$\frac{dy}{dx} = \frac{d}{dx}(20^{\log \tan x})$$

$$= 20^{\log \tan x} . \log 20. \frac{d}{dx} [\log (\tan x)]$$

$$= 20^{\log \tan x} . \log 20. \frac{1}{\tan x} . \frac{d}{dx} (\tan x)$$

$$= 20^{\log \tan x} . \log 20. \frac{1}{\tan x} . \text{Sec}^2 x$$

5) Find the derivative of log [x + $\sqrt{x^2+1}$] with respected to ' x '

Sol : let y = log [x + $\sqrt{x^2+1}$]

$$\frac{dy}{dx} = \frac{d}{dx} [x + \sqrt{x^2+1}]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \frac{d}{dx} [x + \sqrt{x^2+1}]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left[1 + \frac{1}{2\sqrt{x^2+1}} \frac{d}{dx} (x^2+1) \right]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left[1 + \frac{1}{2\sqrt{x^2+1}} . 2x \right]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}} \right]$$

$$= \frac{1}{\sqrt{x^2+1}}$$

6) Find the derivative of $\sqrt{Cos\sqrt{x}}$ with respected to ' x '

Sol : let y = $\sqrt{Cos\sqrt{x}}$

$$\frac{dy}{dx} = \frac{d}{dx} \sqrt{Cos\sqrt{x}}$$

$$= \frac{1}{2\sqrt{Cos\sqrt{x}}} \frac{d}{dx} (Cos\sqrt{x})$$

$$= \frac{1}{2\sqrt{Cos\sqrt{x}}} (- Sin \sqrt{x}) . \frac{d}{dx} (\sqrt{x})$$

$$= \frac{1}{2\sqrt{Cos\sqrt{x}}} (- Sin \sqrt{x}) . \frac{1}{2\sqrt{x}}$$

$$= \frac{-Sin\sqrt{x}}{4\sqrt{x}\sqrt{Cos\sqrt{x}}}$$

7) Find the derivative of x$\sqrt{Sin\ x}$ with respected to ' x '

Sol : let y = x. $\sqrt{Sin\ x}$

$$\frac{dy}{dx} = \frac{d}{dx}\ [x.\ \sqrt{Sin\ x}\ ]$$

$$= x.\ \frac{d}{dx}\sqrt{Sin\ x} + \sqrt{Sin\ x}\ .\ \frac{d}{dx}(x)$$

$$= x.\ \frac{1}{2\sqrt{Sin\ x}}\ \frac{d}{dx}(\text{Sin X})\ .\ \sqrt{Sin\ x}\ .\ 1$$

$$= x.\ \frac{1}{2\sqrt{Sin\ x}}\ .\ \text{Cos X} + \sqrt{Sin\ x}$$

$$= \frac{x\ Cos\ x + 2\ Sin\ x}{2\sqrt{Sin\ x}}$$

7) Find the derivative of $\tan^2\left(\frac{1+x}{1-x}\right)$ with respected to ' x '

Sol : let y = $\tan^2\left(\frac{1+x}{1-x}\right)$

$$\frac{dy}{dx} = \frac{d}{dx}\ \tan^2\left(\frac{1+x}{1-x}\right)$$

$$= \frac{d}{dx}\left[\tan\left(\frac{1+x}{1-x}\right)\right]^2$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ \frac{d}{dx}\ \tan\left(\frac{1+x}{1-x}\right)$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right)\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right).\ \frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= 2.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right).\ \frac{(1-x).1-(1+x)(-1)}{(1-x)^2}$$

$$= \frac{4}{(1-x)^2}.\ \tan\left(\frac{1+x}{1-x}\right).\ Sec^2\left(\frac{1+x}{1-x}\right)$$

Exercise :

I

(i) Cos (4x +7) (ii) e^{ax+b} (iii) $(x^2-3x+5)^{7/3}$ (iv) log (2x - 1) (v) a^{4+5x} (vi) $\sqrt{2+3x}$

(vii) $Cos^n x$ (viii) log (tan 5x) (ix) $\log\left(Sin\frac{x}{2}\right)$

II

(i) log (x^2 +1) (ii) $(x+\sqrt{x^2+1})^n$ (iii) Sin $(3\sqrt{x^2+1})$ (iv) $\sqrt{x+\frac{1}{x}}$ (v) $\frac{1}{\sqrt{2+3x}}$

(vi) $\log\left[Cot(1-x^2)\right]$

III

(i) Sin mx . Cos nx (ii) x^2. Cos^3 2x (iii) $(2x^2+1)\,e^x$ (iv) e^{5x}. log 6x (v) (x^2+1) log (log x)

(vi) e^{2x}. Sin 3x .Cos 4x (vii) Cos $\left[\frac{1-x^2}{1+x^2}\right]$

SUCCESSIVE DIFFERENATION :

If the function in 'x' is differentiated with respect to ' x' , the result is called first derivative at the function. If the 1st derivative is differentiative, the result is called second derivative. Similarly we proced n times, then result is called n^{th} derivative of function.

Let y = f (x) is a function, then first derivative $\frac{dy}{dx} = f^1(x)$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f^{II}(x)$$

Similarly n^{th} derivative

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^{\,n}(x)$$

Solved examples :

1) Find the second derivative of $x^3 + 2x^2 + 3x + 4$ with respected to ' x '

Sol : let $y = x^3 + 2x^2 + 3x + 4$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 2x^2 + 3x + 4)$$

$$= 3x^2 + 4x + 3$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(3x^2 + 4x + 3)$$

$$= (6x + 4)$$

2) Find the second derivative of e^{2x} with respected to ' x '

Sol : let $y = e^{2x}$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = e^{2x}\frac{d}{dx}(2x)$$

$$= e^{2x} \cdot 2$$

$$= 2.\, e^{2x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(2\, e^{2x})$$

$$= 2\frac{d}{dx}(e^{2x})$$

$= 2.2\ e^{2x}$

$= 4.\ e^{2x}$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2}$$

$$= \frac{d}{dx}(4\ e^{2x})$$

$$= 4\ \frac{d}{dx}(e^{2x})$$

$= 4.2\ e^{2x}$

$= 8.\ e^{2x}$

15.4 EXERCISE :

I Find second derivative of

(i) x^4 (ii) Sin 3x (iii) tan x (iv) log x (v) $\frac{1}{x^2}$ (vi) x. log x (vii) e^x . sin x (viii) $x^2\ e^x$

II Find third derivative of

(i) $x^4 + 2x^3 + 7x + 5$ (ii) Cos x (iii) e^{5x} (iv) $\frac{1}{x}$ (v) $e^x\ x^3$

PARTIAL DIFFERENTIATION :

Partial Derivatives :

Let Z = f (x, y) Then $\underset{h\to 0}{Lt}\ \frac{f(a+h,b)-f(a,b)}{h}$ If it exist, is said to be the partial derivative of f (x,y) w.r. to 'x' at (a,b) and is denoted by $\left(\frac{\partial z}{\partial x}\right)$ or f_x (a,b).

If f (x,y) possesses a partial derivative with respect to x at every point of its domain, then the partial derivative with respect to x at every point of its domain, then the partial derivative of f (x,y) w.r. to x is $\underset{h\to 0}{Lt}\ \frac{f(x+h,y)-f(x,y)}{h}$ and is denoted by f_x (x, y) or $\left(\frac{\partial f}{\partial x}\right)$

Note 1 : Partial derivative of z w.r. to 'x' is the ordinary derivative of z w.r. to 'x' treating y as constant . Also partial derivative of z w.r to ' y' is the ordinary derivative of z w.r. to 'y' treating x as constant.

Note 2 : $\left(\frac{\partial z}{\partial x}\right)$ is called first order partial derivative of z w.r to 'x' and are some times denoted by z_x

Examples :

1) Find $\frac{\partial z}{\partial x}$ when $z = x^y + y^x$

Sol : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^y + y^x)$

$$\frac{dy}{dx} = \frac{\partial}{\partial x}(x^y) + \frac{\partial}{\partial x}(y^x)$$

$$= y\, x^{y-1} + y^x \log y$$

2) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ when $z = e^{ax} \sin by$

Sol : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^{ax} \sin by)$

$$= \text{Sin } by \frac{\partial}{\partial x}(e^{ax})$$

$$= \text{Sin } by \,.\, e^{ax} \frac{\partial}{\partial x}(ax)$$

$$= \text{Sin } by \,.\, e^{ax} \,.\, a$$

$$= a.\, e^{ax} .\text{Sin } by$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(e^{ax} \sin by)$$

$$= e^{ax} . \frac{\partial}{\partial y}(\text{Sin } by)$$

$= e^{ax} \text{Cos } by . \dfrac{\partial}{\partial x} (by)$

$= e^{ax} \text{Cos } by . \ b$

$= b. e^{ax} \text{Cos } by$

3) If $u = f(x^2 + y^2)$ prove that $y \dfrac{\partial u}{\partial x} = x \dfrac{\partial u}{\partial y}$

Sol : $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial x} f(x^2 + y^2)$

$= f^{I}(x^2 + y^2) \dfrac{\partial}{\partial x} (x^2 + y^2)$

$= f^{I}(x^2 + y^2). 2x$

$y \dfrac{\partial u}{\partial y} = 2xy \ f^{I}(x^2 + y^2)$

$\dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial x} f(x^2 + y^2)$

$= f^{I}(x^2 + y^2) \dfrac{\partial}{\partial y} (x^2 + y^2)$

$= f^{I}(x^2 + y^2). 2y$

$x \dfrac{\partial u}{\partial y} = 2xy \ f^{I}(x^2 + y^2)$

$\therefore y \dfrac{\partial u}{\partial x} = x \dfrac{\partial u}{\partial y}$

4) If $Z = e^{ax+by} f(ax - by)$ prove that $b \dfrac{\partial z}{\partial x} + a \dfrac{\partial z}{\partial y} = 2 \ abz$

Sol : $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x} [e^{ax+by} f(ax-by)]$

$= e^{ax+by} \frac{\partial}{\partial x} f(ax-by) + f(ax-by) \frac{\partial}{\partial x} e^{ax+by}$

$= e^{ax+by} f^{I}(ax - by) \frac{\partial}{\partial x}(ax - by) + f(ax - by) e^{ax+by} \frac{\partial}{\partial x}(ax + by)$

$= e^{ax+by} f^{I}(ax - by). a + f(ax - by) e^{ax+by}. a$

$b\frac{\partial z}{\partial x} = ab\, e^{ax+by} f^{I}(ax - by) + ab\, e^{ax+by} f(ax - by)$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[e^{ax+by} f(ax - by)]$

$= e^{ax+by} \frac{\partial}{\partial x} f(ax-by) + f(ax-by) \frac{\partial}{\partial x} e^{ax+by}$

$= e^{ax+by} f^{I}(ax - by) \frac{\partial}{\partial x}(ax - by) + f(ax - by) e^{ax+by} \frac{\partial}{\partial x}(ax + by)$

$= e^{ax+by} f^{I}(ax - by). (-b) + f(ax - by) e^{ax+by}. (b)$

$a\frac{\partial z}{\partial x} = -ab\, e^{ax+by} f^{I}(ax - by) + ab .e^{ax+by} . f(ax - by)$

$\therefore\ b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2\, ab .e^{ax+by} . f(ax - by)$

$= 2\, abz$

5) If $z = e^{x+y} + f(x) + g(y)$ find $\frac{\partial^2 z}{\partial x\, \partial y}$

Sol : $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ e^{x+y} + f(x) + g(y)$

$= \frac{\partial}{\partial x}[e^{x+y}] + \frac{\partial}{\partial x}[f(x)] + \frac{\partial}{\partial x}[g(y)]$

$= e^{x+y} + \frac{\partial}{\partial x}(x + y) + 0 + g^{I}(y)$

$= e^{x+y}(1) + g^{I}(y)$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\,[\,e^{x+y} + g^{l}(y)\,]$$

$$= \frac{\partial}{\partial x}\,e^{x+y} + \frac{\partial}{\partial x}\,[\,g^{l}(y)\,]$$

$$= e^{x+y}\,\frac{\partial}{\partial x}(x + y) + 0$$

$$= e^{x+y}\,(1)$$

$$= e^{x+y}$$

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Lesson - 16

CALCULAS - PARTIAL DIFFERENTIATION - MAXIMA - MINIMA

OBJECTIVES:

By the study of this lesson you will be able to understand the meaning, first order partial derivatives, second order partial derivatives in detail.

STRUCTURE:

16.1 INTODUCTION :

Partial Derivation :

Let Z = f(x,y) be a function of two variables in a domain DCR let (a,b) $\in$ D

If $\lim_{h \to o} \frac{f(a+b,b)-f(a,b)}{h}$ exists, then it is said to be the partial derivative of f (x,y) w.r.t x at the point (a,b). It is denoted by $\left[\frac{\partial f}{\partial x}\right]_{(a,b)}$ (or) $\left[\frac{\partial z}{\partial x}\right]_{(a,b)}$ (or) fx (a,b)

If $\lim_{k \to o} \frac{f(a+b,k)-f(a,b)}{k}$ exists, then it is said to be the partial derivative of f(x,y) w.r.t y at the point (a,b). It is denoted by $\left[\frac{\partial f}{\partial y}\right]_{(a,b)}$ (or) $\left[\frac{\partial z}{\partial y}\right]_{(a,b)}$ (or) fy (a,b)

Usually, the partial derivates of Z = f(x,y) at a point (x,y) are denoted by (or) fx and (or)fy

Note :

Partial derivative of z = f(x,y) w.r.t x is the ordinary derivative of z w.r.t x treating y as a constant Partial derivative of z = f (x,y) w.r.t y is the ordinary derivative of z w.r.t y treating x as a constant.

16.2 FRIST ORDER PARTIAL DERIVATIVES :

$\frac{\partial f}{\partial x}$ or fx $\frac{\partial f}{\partial y}$ or fy are called frist order Partial derivatives of z = f(x,y)

16.3 SECOND ORDER PARTIAL DERIVATIVES :

If z = f (x,y) then the frist order partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ may again be functions of x,y.

The partial derivatives of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ w.r.t x and y are called Second order partial derivatives of z = f(x,y)

They are , $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or f x *x*

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\,\partial x} \text{ or f y x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\,\partial y} \text{ or f x y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or f y y}$$

Note : We consider the functions for which $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$

16.4 SOLVED EXAMPLES :

Examples 1 :

If $z = xe^y + ye^x$ then find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

Solution : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(xe^y + ye^x)$

$$= \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial x}(ye^x)$$

$$= e^y\frac{\partial}{\partial x}(x) + y\frac{\partial}{\partial x}(e^x)$$

$$= e^y(1) + y(e^x) = e^y + y\,e^x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(xe^y + ye^x)$$

$$= x\frac{\partial}{\partial x}(e^y) + e^x\frac{\partial}{\partial x}(y)$$

$$= x\,e^y + e^x\,(1) = xe^y + e^x$$

Examples 2 :

If $z = x^2+y^2 + 2xy$ then find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

Solution : $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2 + 2xy)$

$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(2xy)$

$= 2x + 0 + 2y \frac{\partial}{\partial x}(x)$

$= 2x + 2y (1)$

$= 2x + 2y$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2 + 2xy)$

$= \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(2xy)$

$= 0+ 2y + 2x\frac{\partial}{\partial x}(y)$

$= 2y + 2x (1)$

$= 2x + 2y$

$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

Examples 3 :

If $f = x^y$ find $f\,y_{y}$

Solution : $fy = \frac{\partial f}{\partial y}$

$= \frac{\partial}{\partial y}(x^y) = x^y \log x$

$$fy_y = \frac{\partial f}{\partial y} \ (fy)$$

$$= \frac{\partial}{\partial y} \ (x^y \log x)$$

$$= \log x \ \frac{\partial}{\partial y} \ (x^y)$$

$$= \log x \cdot x^y \cdot \log x$$

$$= (\log x)^2 \cdot x^y$$

Examples 4 :

If $f = x^2yz - 2xz^3 + xz^2$ then find $\frac{\partial^2 f}{\partial x\,\partial y}$ at (1,0,2)

Solution : $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2yz - 2xz^3 + xz^2)$

$$= \frac{\partial}{\partial y}(x^2 yz) - \frac{\partial}{\partial y}(2xz^3) + \frac{\partial}{\partial y}(xz^2)$$

$$= x^2 \frac{\partial}{\partial y}(y) - 0 + 0$$

$$= x^2 z (1)$$

$$= x^2 z$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(x^2 z)$$

$$= z \frac{\partial}{\partial x}(x^2)$$

$$= z(2x) = 2xz$$

$$\therefore \frac{\partial^2 f}{\partial x\,\partial y} \text{ at } (1,0,2) = 2(1)(2) = 4$$

Examples 5 :

If $u = \log(x^2 + y^2 + z^2)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution : $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\log(x^2 + y^2 + z^2)\right)$

$$= \frac{1}{x^2 + y^2 + z^2}\frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$= \frac{1}{x^2 + y^2 + z^2}(2x)$$

$$x\frac{\partial u}{\partial x} = x.\frac{1}{x^2 + y^2 + z^2}(2x)$$

$$= \frac{2x^2}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\log(x^2 + y^2 + z^2)\right)$$

$$= \frac{1}{x^2 + y^2 + z^2}\frac{\partial}{\partial y}(x^2 + y^2 + z^2)$$

$$= \frac{1}{x^2 + y^2 + z^2}(2y)$$

$$= \frac{2y}{x^2 + y^2 + z^2}$$

$$y\frac{\partial u}{\partial y} = y.\frac{2y}{x^2 + y^2 + z^2}$$

$$= \frac{2y^2}{x^2 + y^2 + z^2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\log (x^2 + y^2 + z^2) \right)$$

$$= \frac{1}{x^2 + y^2 + z^2} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)$$

$$= \frac{1}{x^2 + y^2 + z^2} (2z)$$

$$z \frac{\partial u}{\partial z} = z. \frac{1}{x^2 + y^2 + z^2} (2z)$$

$$= \frac{2z^2}{x^2 + y^2 + z^2}$$

$$\therefore\ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

$$= \frac{2x^2}{x^2 + y^2 + z^2} + \frac{2y^2}{x^2 + y^2 + z^2} + \frac{2z^2}{x^2 + y^2 + z^2}$$

$$= \frac{2x^2 + 2y^2 + 2z^2}{x^2 + y^2 + z^2}$$

$$= \frac{2(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

$$= 2$$

Examples 6 :

If $z = x^3 + y^3 - 3\ axy$ show that $\frac{\partial^2 z}{\partial x\ \partial y} = \frac{\partial^2 z}{\partial y\ \partial x}$

Solution : $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^3 + y^3 - 3\ axy\ )$

$$= 0 + 3y^2 - 3ax$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\,(3y^2 - 3ax)$$

$$= 0 - 3a$$

$$= -3a$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\,(x^3 + y^3 - 3\,axy)$$

$$= 3x^2 + 0 - 3ay$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]$$

$$= \frac{\partial}{\partial y}\,(3x^2 - 3ay)$$

$$= 0 - 3a$$

$$= -3a$$

$$\therefore \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$$

16.5 EXERCISE :

1. If $z = x^2 + y^2$, find $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}$

2. If $u = e^{x-y}$, find $\frac{\partial^2 u}{\partial x^2}\ \frac{\partial^2 u}{\partial x\, \partial y}$

3. If $f = x^2 (y-z) + y^2 (z-x) + z^2 (x-y)$ show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

4. If $u = e^{xy}$, find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

5. If $z = xy\, f\left(\frac{y}{x}\right)$ show that $x\frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$

16.6 HOMOGENEOUS FUNCTION:

Let A $\in$ R^2, f : A $\rightarrow$ R, n $\in$ R. Then the function f (x,y) is said to be a homogeneous function of degree or order n in variables x, y if f (Kx, Ky) = K^n f(x,y).

16.7 EXAMPLES :

Examples 1 :

If f = $\frac{x^3+y^3}{x^2+y^2}$, show that it is homogeneous function of order 1.

Solution : Let f (x,y) = $\frac{x^3+y^3}{x^2+y^2}$

$\therefore$ f (Kx, Ky) = $\frac{(kx^3)+(ky^3)}{(kx^2)+(ky^2)}$

$= \frac{k^3x^3+k^3y^3}{k^2x^2+k^2y^2}$

$= \frac{k^3(x^3+y^3)}{k^2(x^2+y^2)}$

= k. $\frac{x^3+y^3}{x^2+y^2}$

= k^1. f (x,y)

$\therefore$ f is homogeneous function of order 1 .

Examples 2 :

If f (x,y) = $ax^2 + 2hxy + by^2$ show that f is homogeneous function of order 2.

Solution : Let f (x,y) = $ax^2 + 2hxy + by^2$

$\therefore$ f (Kx, Ky) = a (kx^2) + 2h (kx) (ky) + b $(ky)^2$

= a k^2x^2 + 2h kx ky + b k^2y^2

= k^2 ($ax^2 + 2hxy + by^2$)

= k^2. f (x,y)

$\therefore$ f is homogeneous function of order 2 .

16.8 STATEMENT OF EULER'S THEOREM :

If z = f (x,y) is a homogeneous function of degree n, then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \;\forall$ x,y in the domain of the function.

16.9 EXAMPLES :

Examples 1 :

If $f = \frac{x^3+y^3}{x-y}$, show that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$ by Euler's thorem.

Solution : Let $f(x,y) = \frac{x^3+y^3}{x-y}$

$$\therefore f(kx, ky) = \frac{(kx^3)+(ky^3)}{(kx)-(ky)}$$

$$= \frac{k^3x^3+k^3y^3}{kx-ky}$$

$$= \frac{k^3(x^3+y^3)}{k(x-y)}$$

$$= k^2.\frac{x^3+y^3}{x-y}$$

$$= k^2.f(x,y)$$

$\therefore$ f (x,y) is homogeneous function of order 2 .

By Euler's thorem.

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

Examples 2 :

If f = (x,y), = x Cos $\left(\frac{y}{x}\right)$ + y Cos$\left(\frac{x}{y}\right)$ show that x$\frac{\partial f}{\partial x}$ + y$\frac{\partial f}{\partial y}$ = f by Euler's thorem.

Solution : Let f (x,y) = x Cos $\left(\frac{y}{x}\right)$ + y Cos$\left(\frac{x}{y}\right)$

$\therefore$ f (kx, ky) = kx .Cos $\left(\frac{ky}{kx}\right)$ + ky .Cos$\left(\frac{kx}{ky}\right)$

= kx Cos $\left(\frac{y}{x}\right)$ + ky Cos$\left(\frac{x}{y}\right)$

= k $\left[x \text{ Cos}\left(\frac{y}{x}\right) + y \text{ Cos}\left(\frac{x}{y}\right)\right]$

= k^1. f (x,y)

$\therefore$ f (x,y) is homogeneous function of order 1 .

By Euler's thorem.

$$x \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f$$

$$\Rightarrow x \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1f$$

$$\Rightarrow x \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

16.10 EXERCISE :

1. If z = $\frac{x^{1/3} + y^{1/3}}{x^{1/3} + y^{1/3}}$ find x$\frac{\partial z}{\partial x}$ + y$\frac{\partial z}{\partial y}$ = $\frac{z}{6}$ by Euler's theorem

2. If z = $\frac{x^2 y}{x^3 + y^3}$ find x$\frac{\partial z}{\partial x}$ + y$\frac{\partial z}{\partial y}$ = 0 by Euler's theorem

3. If $u = \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ by Euler's theorem.

4. If $u = x^3 + 3x^2y - 2xy^2 + y^3$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$ by Euler's theorem.

16.11 MAXIMA AND MINIMA :

Stationary Value : Let f be a real function which is differentiable at ' a'. If $f^1(a) = 0$ then we say that f(x) is sationary at x =a . (a, f(a)) is called sationary point and f(a) is called sationary value.

16.12 CRITICAL POINTS :

The points at which $f^1(x) = 0$ or $f^1(x)$ does not exists are called critical points of the function f(x).

16.13 RELATIVE MAXIMUM VALUE :

f (x) has relative maximum at x = a there exist $\delta > 0$ such that $f(x) \leq f(a)$ for $a-\delta < x < a + \delta$. f(a) is called relative maximum value.

16.14 RELATIVE MINIMUM VALUE :

f (x) has relative maximum at x = a if there exist $\delta > 0$ such that $f(x) \geq f(a)$ for $a-\delta < x < a + \delta$. f(a) is called relative minimum value.

16.15 EXTREME VALUE :

The points at which a function attains either maximum or minimum are called extreme points or turnins points of the function maximum or minimum values of a function are called extreme values or turning values of the function

Necessary condition for extreme value of function :

If a function f(x) has extreme value f(a) then $f^1(a) = 0$ if it exists.

Sufficient condition for extreme values :

Let f(x) be derivable at x = a and $f^{11}(a)$ exists and is non - zero.

a) $f^1(a) = 0$ and $f^{11}(a) < 0 \Rightarrow x = a$ is a point of relatively maximum.

b) $f^1(a) = 0$ and $f^{11}(a) > 0 \Rightarrow x = a$ is a point of relatively minimum.

16.16 WORKING RULE FOR FINDING MAXIMUM AND MINIMUM VALUES OF A FUNCTION :

Step I : Find $\frac{dy}{dx}$ for the given function y = f(x)

Step II : Find the values at x when $\frac{dy}{dx}$ = 0 let these values a,b,c,......

Step III : Find $\frac{d^2y}{dx^2}$

Step IV : Find x = a in $\frac{d^2y}{dx^2}$

i) if the result is -Ve, the function is maximum at x = a and that maximum value is f(a)

ii) If the result is +ve, the function is minimum at x = a that minimum value is f(a).

Step V : When $\frac{d^2y}{dx^2}$ = 0 for a particular value x = a, then the either employ the first method or find $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ and put x = a successively in these derivatives.

16.17 EXAMPLES :

Examples 1 :

Find the stationary points and stationary values of the function $3x^4 - 4x^3 + 1$.

Solution : Let $y = 3x^4 - 4x^3 + 1$.

$\frac{dy}{dx} = 12x^3 - 12x^2$

$\frac{d^2y}{dx^2} = 36x^2 - 24x$

Put $\frac{dy}{dx} = 0$

$\Rightarrow 12x^3 - 12x^2 = 0$

$\Rightarrow 12x^2(x-1) = 0$

$\Rightarrow x^2 = 0,\ x-1 = 0$

$\Rightarrow x = 0,\ x = 0$

Stationary points 0,1

Stationary values = Y x= 0, Y x=1

$= 3(0)^4 - 4(0)^3 + 1,\ 3(1)^4 - 4(1)^3 + 1$

= 1,0

Examples 2 :

Find the maximum and minimum values of the function $x^3 - 9x^2 + 24x - 12$

Solution : Let $y = x^3 - 9x^2 + 24x - 12$

$$\frac{dy}{dx} = 3x^2 - 18x + 24$$

$$\frac{d^2y}{dx^2} = 6x - 18$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 - 18x + 24 = 0$

$\Rightarrow x^2 - 6x + 8 = 0$

$\Rightarrow x^2 - 2x - 4x + 8 = 0$

$\Rightarrow x(x-2) = 4(x-2) = 0$

$\Rightarrow (x-2)(x-4) = 0$

Stationary Values 2,4

i) When x = 2

$$\frac{d^2y}{dx^2} = 6(2) - 18 = -6 < 0$$

$y = 2^3 - 9.2^2 + 24.2 - 12$

$= 8 - 36 + 48 - 12$

$= 8$

When x = 2 the given function has maximum value and that maximum value is 8.

ii) When x = 4

$\frac{d^2y}{dx^2}$ = 6(4) - 18 = 6>0

y = $4^3 - 9.4^2 + 24.4 - 12$

= 64 - 144 + 96 -12

= 160 - 156

= 4

When x = 4 the given function has minimum value and that minimum value is 4.

Examples 3 :

Show that the function f (x) = $\frac{x}{\log x}$ has a minimum value at x = e.

Solution : f (x) = $\frac{x}{\log x}$

$$f^1(x) = \frac{\log x \frac{d}{dx}(x) - x\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{\log x.1}{(\log x)^2}$$

$$= \frac{\log x - 1}{(\log x)^2}$$

$$f^{11}(x) = \frac{(\log x)^2 \frac{d}{dx}(\log x - 1) - (\log x - 1)\ \frac{d}{dx}(\log x)^2}{(\log x)^4}$$

$$= \frac{(\log x)^2 . \frac{1}{x} - (\log x - 1)\ .2\log x.\frac{1}{x}}{(\log x)^4}$$

$$= \frac{\frac{1}{x}(\log x)^2 . \frac{2}{x}\log x\ .(\log x - 1)}{(\log x)^4}$$

Put $f^1(x) = 0$

$\Rightarrow \frac{\log x - 1}{(\log x)^2} = 0$

$\Rightarrow \log x - 1 = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \log_e x = \log_e e$

$\Rightarrow x = e$

Stationary value = e

When x = e

$$f^{11}(e) = \frac{\frac{1}{e}(\log e)^2 . \frac{2}{e}\log e \;.(\log e - 1)}{(\log e)^4}$$

$$= \frac{\frac{1}{e}1^2 . \frac{2}{e}.1\;(1-1)}{(1)^4}$$

$$= \frac{1}{e} > 0$$

$f(e) = \frac{e}{\log e} = \frac{e}{1} = e$

$\therefore$ f (x) has minimum value at x = e and that minimum value is e

Examples 4 :

Investigate the maxima and minima of the function $2x^3 + 3x^2 - 36x + 10$

Solution : Let $y = 2x^3 + 3x^2 - 36x + 10$

$\frac{dy}{dx} = 6x^2 + 6x - 36$

$\frac{d^2y}{dx^2} = 12x + 6$

Put $\frac{dy}{dx} = 0$

$\Rightarrow 6x^2 + 6x - 36 = 0$

$\Rightarrow x^2 + x - 6 = 0$

$\Rightarrow$ (x+3) (x-2) = 0

Stationary Values -3,2

When x = -3

$\frac{d^2y}{dx^2}$ = 12 (-3) + 6 = -30 < 0

x = - 3

y = 2 $(-3)^3$ + 3 $(-3)^2$ - 36 (-3) +10

= - 54 + 27 + 108 +10

= 91

$\therefore$ The function has maximum value at x = -3 and that maximum value is 91

When x = 2

$\frac{d^2y}{dx^2}$ = 12 (2) + 6 = 30 > 0

y = 2 $(2)^3$ + 3 $(2)^2$ - 36 (2) +10

= 16 +12 -72 +10

= -34

$\therefore$ The function has maximum value at x = 2 and that maximum value is -34

Examples 5 :

A company has examined its cost structure and revenue structure and has determined that c the total cost , R total revenue, and x the number of units produced are related as

C = 100 + 0.015 x^2 and R = 3x

Find the production rate x that will maximise points of the company. Find the profit. Find also the profit when x = 120.

Solution : Let p denote the profit of the company, then

p = R - c

$\Rightarrow$ p = 3x (100 + 0.015 x^2)

= 3x - 100 - $\frac{15}{1000}x^2$

$\frac{dp}{dx}$ = 3 - $\frac{30x}{1000}$

$$\frac{d^2p}{dx^2} = \frac{-30}{1000} < 0$$

Put $\frac{dp}{dx} = 0$

$\Rightarrow 3 - \frac{30x}{1000} = 0$

$\Rightarrow$ x = 100 units

Stationary value x = 100.

When x = 100

$$\frac{d^2p}{dx^2} = \frac{-30}{1000} < 0$$

Put = 3(100) -100 - $\frac{15}{1000}$ $(100)^2$

= 3 -100 - $\frac{15 \text{ x } 100 \text{ x } 100}{1000}$

= 3 -100 - 150

= 50 rupees

When x = 100 the profit is maximum and that maximum profit is 50 rupees .

The profit when x = 120 is

p = 3 x120 - 100 - 0.015 x $(120)^2$

= 360 - 100 - 216

= 44 rupees.

Examples 6 :

The cost C of manu facturing a certain article is given by the formula

$$C = 5 + \frac{48}{x} + 3x^2$$

Where x is the number of articles manufactured. Find minimum value of c.

Solution : $C = 5 + \frac{48}{x} + 3x^2$

$\frac{dc}{dx} = 0 + 48\left(\frac{-1}{x^2}\right) + 6x$

$\frac{d^2c}{dx^2} = 48\left(\frac{2}{x^3}\right) + 6$

Put $\frac{dc}{dx} = 0$

$\Rightarrow \frac{-48}{x^2} + 6x = 0$

$\Rightarrow 6x = \frac{48}{x^2}$

$\Rightarrow 6x^3 = 48$

$\Rightarrow x^3 = 8$

$\Rightarrow x = 2$

Stationary value 2

When x = 2

$\frac{d^2c}{dx^2} = \frac{96}{2^3} + 6$

$= 12 + 6$

$= 18 > 0$

$C = 5 + \frac{48}{2} + 3(2^2)$

$= 5 + 24 + 24$

= 53 rupees

When x = 2 the cost is minimum and that minimum cost is 53 rupees .

16.18 EXERCISES :

Find the maximum (or) minimum values of the function.

1. $y = 27 - 6x + x^2$
2. $y = 8 + 4x - x^2$

Find the maximum and minimum values of the function.

3. $y = \frac{2}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$
4. $y = x^3 - 3x + 15$
5. $y = x^3 - 2x^2 + x + 6$
6. $y = 2x^3 + 3x^2 - 36x + 10$
7. By an ' Economic order Quantity; We mean a quantity Q,Which when purchased in each order, minimizes the total cost T incurred in obtaining and staring material for a certain time period to 10 fulfil a given rate of demand for the material during the time period.

   The material demanded is 10,000 units per year, the cost price of material Re.1 per unit, the cost of replenishing the stock of material per order regardless of the size Q of the order is Rs. 25, and the cost of storing the material is $12\frac{1}{2}$ percent per year on the rupee value of average quantity Q/2 on hand.

   (i) show that $T = 10,000 + \frac{2,50,000}{Q} + \frac{Q}{16}$

   (ii) Find the Economic order quantity and the cost T corresponding to that.

   (iii) Find the total cost when each order is placed for 2500 units.
8. The demand function for a particular commodity is $y = 15\, e^{-x/3}$ for $0 \leq x \leq 8$, where y is the price per unit and 'x' is the number of units demanded. Determine the price and the quantity for which the revenue is maximum.

   (Hint : Revenue ; R = y , x)
9. A firm has to produce 144,000 units of an item per year. It cost Rs. 60 to make the factory ready for a product run of the item regardless of units x produced in a run. The cost of material is Rs. 5 per unit and the cost of storing the material is 50 paiseper item per year on the average inventory $\left(\frac{x}{2}\right)$ in hand. Show that the total cost cis given by

$$C = 720,000 + \frac{23,040,000}{x} + \frac{x}{4}$$

Find also the economic lost size, i.e. value of x for which ' C ' is minimum.

10. A company notices that higher sales, of a particular item which it produces are achieved by lowering the price charged. As a result the total revenue from the sales at first rises as the number of units sold increases, reaches the highest point and then falls off. This pattern of total revenue is described. by the relation $y = 40,00,000 - (x - 2000)^2$ Where y is the total revenue and x the number of units sold.

    i) Find what number of units sold maximizes total revenue ?

    ii) What is the amount of this maximum revenue ?

    iii) What would be the total revenue if 2500 units were sold ?

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Lesson - 17

CALCULAS - INTEGRATION

OBJECTIVES:

By the study of this lesson you will be able to understand the meaning & important of the indefinite integrals with examples. You will also be thorough with integration by substitution, Definite integrats in detail.

STRUCTURE:

17.1 INTODUCTION :

Antiderivation :

If f(x) and g(x) are two functions such that $f^{I}(x) = g(x)$ then f(x) is called antiderivative or primitive of g(x) with respect to x

Ex : $\frac{d}{dx}(x^2 +2x+5) = 2x+2$

$x^2 +2x+5$ is an antiderivative of 2x + 2.

If f(x) is Primitive of f(x) then F(x)+c is called indefinite integral of f(x) with respect to x. It is denoted by $\int f(x)dx$.

$\int f(x)dx$ = F(x)+c where c is constant.

Ex : 1 - $\int\left(x^4+2x\right)dx = \frac{x^5}{5}$ + x^2+c

Ex : 2 - $\int Sinx\ dx =$ - Cos x + c

17.2 THEOREMS :

Theorem 1 :

If n $\neq$ - 1 then $\int x^n\ dx = \frac{x^{n+1}}{n+1}$ + c

Proof :

$$\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\therefore \int x^n\ dx = \frac{x^{n+1}}{n+1} + c$$

Ex : $\int x^7\ dx = \frac{x^{7+1}}{7+1} + c$

$$= \frac{x^8}{8} + c$$

Ex : $\int x^{-6}\ dx = \frac{x^{-6+1}}{-6+1} + c$

$$= \frac{x^{-5}}{-5} + c$$

Ex : $\int x^{2/3}\ dx = \frac{x^{2/3+1}}{2/3+1} + c$

$$= \frac{3}{5}x^{5/3} + c$$

Theorem 2 :

$$\int a\,dx = ax + c$$

Proof :

$$\because \frac{d}{dx}(ax) = a$$

$$\therefore \int a\,dx = ax + c$$

Ex : $\int 2\,dx = 2 + c$

Ex : $\int dx = x + c$

Theorem 3 :

$$\int \frac{1}{x}dx = \log|x| + c$$

Proof :

$$\because \frac{d}{dx}\left(\log|x|\right) = \frac{1}{x}$$

$$\therefore \int \frac{1}{x}\,dx = \log|x| + c$$

Ex : $\int \frac{1}{ax}dx = \frac{1}{a}\int \frac{1}{x}dx = \frac{1}{a}\log|x| + c$

Ex : $\int \frac{1}{5x}dx = \frac{1}{5}\int \frac{1}{x}dx = \frac{1}{5}\log|x| + c$

Theorem 4 : $\int e^x dx = e^x + c$

Proof :

$$\because \frac{d}{dx}\left(e^x\right) = e^x$$

$$\therefore \int e^x\,dx = e^x + c$$

Theorem 5 : $\int \frac{1}{x^n}\,dx = \frac{1}{(n-1)x^{n-1}} + c$

Proof :

$$\because \frac{d}{dx}\left[\frac{-1}{(n-1)x^{n-1}}\right]$$

$$= \frac{d}{dx}\left[\frac{-1}{(n-1)}.x^{n-1}\right]$$

$$= \frac{-1}{(n-1)}\left[-(n-1)\right]x^{-(n-1)-1}$$

$$= x^{-n+1-1}$$

$$= \frac{1}{x^n}$$

$$\therefore \int \frac{1}{x^n}\,dx = \frac{1}{(n-1)x^{n-1}} + c$$

Ex : $\int \frac{1}{x^5}\,dx = \frac{-1}{(5-1)x^{5-1}} + c$

$$= \frac{-1}{4x^4} + c$$

Ex : $\int \frac{1}{x^{5/3}}\,dx = \frac{-1}{\left(\frac{5}{3}-1\right)x^{(5/3-1)}} + c$

$$= \frac{-1}{\frac{2}{3}x^{2/3}} + c$$

Theorem 6 : $\int a^x dx = \frac{a^x}{\log a} + c$

Proof :

$$\because \frac{d}{dx}\left[\frac{a^x}{\log a}\right] = \frac{1}{\log a}\frac{d}{dx}(a^x)$$

$$= \frac{1}{\log a}.\ a^x.\ \log a$$

$$\therefore \int a^x dx = \frac{a^x}{\log a} + c$$

Ex : $\int 2^x dx = \frac{2^x}{\log 2} + c$

Ex : $\int \left(\frac{5}{2}\right)^x dx = \frac{\left(\frac{5}{2}\right)^x}{\log\left(\frac{5}{2}\right)} + c$

Theorem 7 : $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

Proof :

$$\because \frac{d}{dx}\left[2\sqrt{x}\right] = 2\frac{d}{dx}(\sqrt{x})$$

$$= 2.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}$$

$$\therefore \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$$

Ex : $\int \frac{1}{\sqrt{5x}}\,dx = \int \frac{1}{\sqrt{5}\sqrt{x}}\,dx$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{x}}\,dx$$

$$= \frac{1}{\sqrt{5}}\,2\sqrt{x} + c$$

$$= \frac{2}{\sqrt{5}}\sqrt{x} + c$$

Ex : $\int \frac{1}{\sqrt{\frac{x}{7}}}\,dx = \int \sqrt{\frac{7}{x}}\,dx$

$$= \int \frac{\sqrt{7}}{\sqrt{x}}\,dx$$

$$= \sqrt{7}\int \frac{1}{\sqrt{x}}\,dx$$

$$= \sqrt{7}.2\sqrt{x} + c$$

$$= 2\sqrt{7}\sqrt{x} + c$$

Theorem 8 : $\int 0.dx = c$

Proof :

$$\because \frac{d}{dx}(c) = 0$$

$$\therefore \int 0.dx = 0 + c = c$$

Theorem 9 : $\int \text{Cos x dx} = \text{Sin x} + c$

Proof :

$$\because \frac{d}{dx}(Sin\ x) = Cos\ x$$

$$\therefore \int \text{Cos x dx} = \text{Sin x} + c$$

Theorem 10 : $\int \text{Sin x dx} = -\text{Cos x} + c$

Proof :

$$\because \frac{d}{dx}(-Cos\ x) = -\left[\frac{d}{dx}(Cos\ x)\right]$$

= - (-Sin x) = Sin x

$$\therefore \int \text{Sin x dx} = -\text{Cos x} + c$$

Theorem 11 : $\int \text{Sec}^2 \text{ x dx} = \tan x + c$

Proof :

$$\because \frac{d}{dx}(\tan\ x) = Sec^2 x$$

$$\therefore \int \text{Sec}^2 \text{ x dx} = \tan x + c$$

Theorem 12 : $\int \text{Cosec}^2 \text{ x dx} = -\text{Cot x} + c$

Proof :

$$\because \frac{d}{dx}(-Cot\ x) = -\left[\frac{d}{dx}(Cot\ x)\right]$$

$= -[-\text{Cosec}^2 x]$

$= \text{Cosec}^2 x$

$$\therefore \int \text{Cosec}^2 \text{ x dx} = -\text{Cot x} + c$$

Theorem 13 : $\int \text{Sec tan x dx} = \text{Sec x} + c$

Proof :

$$\because \frac{d}{dx}\left(Sec\ x\right) = Sec\ x \tan x$$

$$\therefore \int \text{Sec tan x dx} = \text{Sec x} + c$$

Theorem 14 : $\int \text{CoSec x Cot x dx} = -\text{Cosec x} + c$

Proof :

$$\because \frac{d}{dx}\left(-Cosec\ x\right) = -\left[\frac{d}{dx}\left(Cosec\ x\right)\right]$$

$$= -[-\text{Cosec x Cot x}]$$

$$= \text{Cosec x Cot x}$$

$$\therefore \int \text{CoSec x Cot x dx} = -\text{Cosec x} + c$$

17.3 SOLVED EXAMPLES :

Examples 1 : Evaluate $\int (3x^2 - 5x + 4)\, dx$

Solution : $\int (3x^2 - 5x + 4)\, dx$

$$= 3\int x^2\, dx - 5\int x\, dx + \int u\, dx$$

$$= 3\frac{x^{2+1}}{2+1} - 5.\frac{x^{1+1}}{1+1} + 4x + c$$

$$= x^3 - \frac{5}{2}x^2 + 4x + c$$

Examples 2 : Evaluate $\int\left(\frac{3x+4}{2\sqrt{x}}\right)^2 dx$

Solution : $\int\left(\frac{3x+4}{2\sqrt{x}}\right)^2 dx$

$$= \int \frac{9x^2+24x+16}{4x} dx$$

$$= \int\left(\frac{9}{4}x^2+12x+\frac{4}{x}\right)dx$$

$$= \frac{9}{4}\int x^2\,dx + 12\int x\,dx + 4\int \frac{1}{x}dx$$

$$= \frac{9}{4}\frac{x^{2+1}}{2+1} + 12\frac{x^{1+1}}{1+1} + 4\log x + c$$

$$= \frac{3}{4}x^3 + 6x^2 + 4\log x + c$$

Examples 3 : Evaluate $\int(1-x)(4-3x)dx$

Solution : $\int(1-x)(4-3x)dx$

$$= \int\left(4-7x+3x^2\right)dx$$

$$= \int 4dx - 7\int x\,dx + 3\int x^2 dx$$

$$= 4x. - 7\frac{x^{1+1}}{1+1} + 3.\frac{x^{2+1}}{2+1} + c$$

$$= 4x - \frac{7}{2}x^2 + x^3 + c$$

Examples 4 : Evaluate $\int\left(\frac{3x^2+4x+5}{\sqrt{x}}\right)dx$

Solution : $\int\left(\frac{3x^2+4x+5}{\sqrt{x}}\right)dx$

$$= \int\left(\frac{3x^2}{\sqrt{x}}+\frac{4x}{\sqrt{x}}+\frac{5}{\sqrt{x}}\right)dx$$

$$= \int\left(3x^{3/2}+4\sqrt{x}+\frac{5}{\sqrt{x}}\right)dx$$

$$= 3\int x^{3/2}\,dx+4\int\sqrt{x}\,dx+5\int\frac{1}{x}\,dx$$

$$= 3\frac{x^{3/2}+1}{\frac{3}{2}+1}+4\frac{x^{1/2}+1}{\frac{1}{2}+1}+5\,.2\sqrt{x}+c$$

$$= 3.\ \frac{2}{5}x^{\frac{5}{2}}+4\ \frac{2}{3}x^{\frac{3}{2}}+10\sqrt{x}+c$$

$$= \frac{6}{5}x^{\frac{5}{2}}+\frac{8}{3}x^{\frac{3}{2}}+10\sqrt{x}+c$$

Examples 5 : Evaluate $\int\left(\frac{a^x - b^x}{a^x.b^x}\right)^2 dx$

Solution : $\int\left(\frac{a^x - b^x}{a^x.b^x}\right)^2 dx$

$$= \int\frac{a^{2x} - 2a^xb^x + b^{2x}}{a^x.\,b^x}dx$$

$$= \int \left(\frac{a^{2x}}{a^x . b^x} - \frac{2a^x b^x}{a^x . b^x} + \frac{b^{2x}}{a^x . b^x} \right) dx$$

$$= \int \left(\frac{a^{2x}}{b^x} - 2 + \frac{b^x}{a^x} \right) dx$$

$$= \int \left(\frac{a}{b} \right)^x dx - \int 2\, dx + \int \left(\frac{b}{a} \right)^x dx$$

$$= \frac{\left(\frac{a}{b} \right)^x}{\log \left(\frac{a}{b} \right)} - 2x + \frac{\left(\frac{b}{a} \right)^x}{\log \left(\frac{b}{a} \right)} + c$$

Examples 6 : Evaluate $\int (e^x + \sin x + \sqrt{x}\ ) dx$

Solution : $\int (e^x + \sin x + \sqrt{x}\ ) dx$

$$= \int e^x dx + \int \sin x\, dx + \int \sqrt{x}\, dx$$

$$= \int e^x - \text{Cos}\, x + \frac{x^{1/2} + 1}{\frac{1}{2} + 1} + c$$

$$= e^x - \text{Cos}\, x + \frac{2}{3} x^{3/2} + c$$

Examples 6 : Evaluate $\int (\text{Cosec}^2 x - \text{Cos}\, x + \frac{1}{x}) dx$

Solution : $\int (\text{Cosec}^2 x - \text{Cos}\, x + \frac{1}{x}) dx$

$$= \int \text{Cosec}^2 x\, dx - \int \text{Cos}\, x\, dx + \int \frac{1}{x} dx$$

$$= \int \text{Cosec}^2 x\,dx - \int \text{Cos}\,x\,dx + \int \frac{1}{x}\,dx$$

$$= \int -\text{Cot}\,x - \text{Sin}\,x + \log x + c$$

Examples 7 : Evaluate $\int (8e^x - 4a^x + 3\cos x + \sqrt[4]{x})\,dx$

Solution : $\int (8e^x - 4a^x + 3\cos x + \sqrt[4]{x})\,dx$

$$= \int 8e^x\,dx - \int 4a^x dx + \int 3\cos x\,dx + \int x^{1/4}\,dx$$

$$= 8\int e^x\,dx - 4\int a^x dx + 3\int \cos x\,dx + \int x^{1/4}\,dx$$

$$= 8e^x - 4\frac{a^x}{\log a} + 3\sin x + \frac{x^{1/4+1}}{\frac{1}{4}+1} + c$$

$$= 8e^x - \frac{4a^x}{\log a} + 3\sin x + \frac{4}{5}x^{\frac{5}{4}} + c$$

Examples 8 : Evaluate $\int \left(\frac{a + b\sin x}{\cos^2 x}\right) dx$

Solution : $\int \left(\frac{a + b\sin x}{\cos^2 x}\right) dx$

$$= \int \left(\frac{a}{\cos^2 x} + \frac{b\sin x}{\cos^2 x}\right) dx$$

$$= \int (a\sec^2 x + b\sec x\tan x)\,dx$$

$$= a\int \sec^2 x + b\int \sec x\tan x\,dx$$

$$= a\tan x + b\sec x + c$$

Examples 9 : Evaluate $\int \frac{\sin x}{1+\sin x} dx$

Solution : $\int \frac{\sin x}{1+\sin x} dx$

$$= \int \frac{\sin x\,(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= \int \frac{\sin x - \sin^2 x}{1-\sin^2 x} dx$$

$$= \int \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx$$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x} \right) dx$$

$$= \int (\sec x \tan x - \tan^2 x) dx$$

$$= \int \sec x \tan x\, dx - \int (\sec^2 x - 1)\, dx$$

$$(\because \sec^2 x \tan x = 1)$$

$$= \int \sec x \tan x\, dx - \int \sec^2 x\, dx + \int 1\, dx$$

$$= \text{Sec } x - \tan x + x + c$$

17.4 EXERCISE - 1 :

Evaluate

1. $\int 3x^4\, dx$

2. $\int 7x^{3/2}\, dx$

3. $\int \left(2x^{3/2} + 4x + 5\right) dx$

4. $\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 dx$

5. $\int\left(1+\frac{2}{x}+\frac{3}{x^2}\right)dx$

6. $\int\left(\frac{x^3+2x^2-4x+5}{x^2}\right)dx$

7. $\int\frac{(2x+3)^2}{3x}dx$

8. $\int(2x+3)(4x-1)\,dx$

9. $\int\left(1-x^2\right)^2 dx$

10. $\int\left(x-\frac{1}{x}\right)^3 dx$

11. $\int\sqrt[5]{x^3}\,dx$

12. $\int\left(\frac{3}{\sqrt{x}}-\frac{2}{x}+\frac{1}{3x^2}\right)dx$

13. $\int\left(\frac{1-\sqrt{x}}{x}\right)dx$

14. $\int\left(\frac{1}{\sqrt{x}}+\frac{7}{x^{3/2}}+\frac{3}{2x^2}\right)dx$

15. $\int\left(\frac{2\cos x}{5\sin^2 x}+\frac{1}{5\cos^2 x}\right)dx$

16. $\int\sqrt{1+\sin 2x}\,dx$

17. $\int\left(2x+\frac{1}{2}e^{-x}+\frac{4}{x}-\frac{1}{\sqrt[3]{x}}\right)dx$

17.5 INTEGRATION BY SUBSTITUTION :

Theorem : If $\int f(x)\,dx = g(x)$ and $a \neq 0$ then $\int f(ax+b)\,dx = \frac{1}{a}$

Proof :

Put ax + b = t

$\Rightarrow \frac{d}{dx}(ax+b) = \frac{d}{dx}(t)$

$\Rightarrow a+0 = \frac{dt}{dx}$

$\Rightarrow a.dx = dt$

$\Rightarrow dx = \frac{1}{a}dt$

$\therefore \int f(ax+b)\,dx$

$= \int f(t) . \frac{1}{a}dt$

$= \frac{1}{a}\int f(t)\,dt$

$= \frac{1}{a}\int g(t) + c$

$= \frac{1}{a} g(ax+b) + c$

Theorem : $\int \tan dx = \log|\sec x| + c$

Proof : $\int \tan x \; dx = \int \frac{\sin x}{\cos x}\,dx$

Put cos x = t Differentiate

- sin x dx = dt

sin dx = - dt

$= \int \frac{-dt}{t}$

$= -\log(t) + c$

$= \log\left(\frac{1}{t}\right) + c$

$= \log\left(\frac{1}{\text{Cos } x}\right) + c$

$= \log|\sec x| + c$

Theorem : $\int \cot dx = \log|\sin x| + c$

Proof : $\int \cot dx = \int \frac{\cos x}{\sin x} dx$

Put $\sin x = t$ Differentiate

$\cos x \, dx = dt$

$= \int \frac{dt}{t}$

$= \log(t) + c$

$= \log|\sin x| + c$

Theorem : $\int \sec x \, dx = \log|\sec x - \tan x|$

Proof : $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

Put $\sec x - \tan x = t$ Differentiate

$(\sec x \tan x + \sec^2 x) \, dx = dt$

$\sec x(\tan x + \sec x) \, dx = dt$

$= \int \frac{dt}{t}$

$= \log(t) + c$

$= \log|\sec x + \tan x| + c$

Theorem : $\int \text{cosec } x \, dx = \log|\text{cosec } x - \cot x| + c$

Proof : $\int \text{cosec } x \, dx = \int \frac{\text{cosec } x (\text{cosec } x - \cot x)}{\text{cosec } x - \cot x} dx$

Put cosec x - cot x = t Differentiate

$(-\text{cosec } x \cot x + \text{cosec}^2 x) \, dx = dt$

$\text{cosec } x (\text{cosec } x - \cot x) \, dx = dt$

$= \int \frac{dt}{t}$

$= \log(t)$

$= \log|\text{cosec } x - \cot x| + c$

17.6 SOLVED EXAMPLES :

Examples 1 : Evaluate $\int \text{Sin}(2x+3) \, dx$

Sol : Put 2x + 3 = t Differentiate

$(2 + 0) \, dx = dt$

$dx = \frac{dt}{2}$

$\therefore \int \text{Sin}(2x+3) \, dx$

$= \int \text{Sin } t \frac{dt}{2}$

$= \frac{1}{2} \int \text{Sin } t \, dt$

$= \frac{1}{2}(-\cos t) + c$

$= -\frac{1}{2} \cos(2x+3) + c$

Examples 2 : Evaluate $\int \frac{dx}{\sqrt{1+5x}}$

Sol : Put 1 + 5x = t Differentiate

$$(0 + 5)\ dx = dt$$

$$dx = \frac{dt}{5}$$

$$\therefore \int \frac{dx}{\sqrt{1+5x}}$$

$$= \int \frac{\frac{dt}{5}}{\sqrt{t}}$$

$$= \frac{1}{5}\int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{5} 2.\sqrt{t} + c$$

$$= \frac{2}{5}\sqrt{1+5x} + c$$

Examples 3 : Evaluate $\int \frac{e \log x}{x}\, dx$

Sol : Put log x = t Differentiate

$$\frac{1}{x}\ dx = dt$$

$$\therefore \int \frac{e \log x}{x}\, dx$$

$$= \int e^{t}\ dt$$

$$= \int e^{t} + c$$

$$= \int e^{\log x} + c$$

Examples 4 : Evaluate $\int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx$

Sol : Put log $\sqrt{x}$ = t Differentiate

$$\frac{1}{2\sqrt{x}}\, dx = dt$$

$$\frac{1}{\sqrt{x}}\, dx = 2.\, dt$$

$$\therefore \int \frac{1}{\sqrt{x}} \cos \sqrt{x}\, dx$$

$$= \int \cot t.\, 2\, dt$$

$$= 2\int \cot t\, dt$$

$$= 2 \sin t + c$$

$$= 2 \sin \sqrt{x} + c$$

Examples 5 : Evaluate $\int \sec(\tan x) \sec^2 x\, dx$

Sol : Put log tan x = t Differentiate

$$\sec^2 dx = dt$$

$$\therefore \int \sec(\tan x) \sec^2 x\, dx$$

$$= \int \sec t.\ dt$$

$$= \log|\sec t + \tan t| + c$$

$$= \log|\sec(\tan x) + \tan(\tan x)| + c$$

Examples 6 : Evaluate $\int \frac{\text{cosec}^2 x}{a+b\cot x}\,dx$

Sol : Put a+b cot x = t Differentiate

$\Rightarrow [\,0 + b\,(-\text{cosec}^2 x)\,]\; dx = dt$

$\Rightarrow -b\;\text{cosec}^2 x\;\; dx = dt$

$\Rightarrow \text{cosec}^2 x\;\; dx = \frac{dt}{-b}$

$\therefore \int \frac{\text{cosec}^2 x}{a+b\cot x}\,dx$

$= \int \frac{\frac{dt}{-b}}{t}$

$= \frac{-1}{b}\int \frac{dt}{t}$

$= \frac{-1}{b}\log|\,t\,| + c$

$= \frac{-1}{b}\log|\,a+b\cot x\,| + c$

Examples 7 : Evaluate $\int \frac{2x+3}{\sqrt{x^2+3x-4}}\,dx$

Sol : Put $x^2 + 3x - 4 = t$ Differentiate

$(2x+3)\,dx = dt$

$\therefore \int \frac{2x+3}{\sqrt{x^2+3x-4}}\,dx$

$= \int \frac{dt}{\sqrt{t}}$

$= 2\sqrt{t} + c$

$= 2\sqrt{x^2+3x-4} + c$

Examples 8 : Evaluate $\int \left[\frac{\sec x}{(1-\tan x)^2} \right]^2 dx$

Sol : Put 1- tan x = t Differentiate

$(0 - \sec^2 x)\, dx = dt$

$\sec^2 x \;\; dx = -\, dt$

$$\therefore \int \left[\frac{\sec x}{(1-\tan x)^2} \right]^2 dx$$

$$= \int \frac{\sec x}{(1-\tan x)^4} dx$$

$$= \int \frac{-dt}{t^4}$$

$$= \int \frac{dt}{t^4}$$

$$= -\left[\frac{-1}{3\,t^3} \right] + c$$

$$= \frac{1}{3\,t^3} + c$$

$$= \frac{1}{3\,(1-\tan x)^3} + c$$

17.7 EXERCISE - 2 :

1. $\int \frac{1}{5x-6} dx$
2. $\int (7x-4)^{3/4} dx$
3. $\int \frac{dx}{\sqrt{11-5x}}$
4. $\int \frac{6x}{3x^2-2} dx$

5. $\int \frac{2x - 3}{x^2 - 3x + 4}\, dx$

6. $\int \frac{1}{x \log x}\, dx$

7. $\int x^3 . \sin^4\, dx$

8. $\int 2x . e^{x^2}\, dx$

9. $\int 2x . \cos(1 + x^2)\, dx$

10. $\int e^x . \sin(e^x)\, dx$

11. $\int \left(3x^2 - 4\right) x\, dx$

12. $\int \frac{1}{1 + (2x + 1)}\, dx$

13. $\int \frac{\log x}{x}\, dx$

14. $\int \frac{\sin\left(\log x\right)}{x}\, dx$

15. $\int \frac{\cos\left(\log x\right)}{x}$

16. $\int \frac{\log\left(1 + x\right)}{1 + x}\, dx$

17. $\int \frac{\left(1 + \log x\right)^n}{x}\, dx$

18. $\int \cos^3 x \sin x\, dx$

19. $\int \sqrt[3]{\sin x} . \cos x\, dx$

20. $\int \tan^5 x . \sec^2 x\, dx$

20. $\int \tan^5 x . \sec^2 x \, dx$

21. $\int \text{cosec}^5 x . \sqrt{\cot x} \, dx$

22. $\int \frac{\cos x}{(1+\sin x)^2} dx$

23. $\int \frac{\sec^2 x}{(1+\tan x)^4} dx$

24. $\int \frac{\text{cosec}^2 x}{(a+b\cot x)^6} dx$

17.8 DEFINITE INTEGRALS :

Upper and Lower Bounds of a function :

Let s be a subset of the domain of f. If there exists a number M such that $f(x) \leq M$ for every x in s, then we say that f is bounded above and M is called an upper bound. If there exists a number m such that $m \leq f(x)$ for every x in S, then we say that f is bounded below and m is called a lower bound. If a function is both bounded above and below then we say that f is bounded.

17.9 DEFINIT INTEGRAL AS THE LIMIT OF A SUN :

Definition :

Let f be a bounded function in an interval [a,b] contained in the domain of f. Let $x_0, x_1, x_2, \ldots\ldots, x_{r-1}, x_r, \ldots\ldots x_{n-1}, x_n$ be numbers such that

$$a = x_0 < x_1 < x_2 < \ldots\ldots, x_{r-1} < x_r < \ldots\ldots < x_{n-1} < x_n = b$$

Let δ_r be the length of the r[th] interval. $[x_{r-1}, x_r]$ $\delta_r = x_r - x_{r-1}$ m_r and M_r are the lower and upper bounds in the r[th] interval.

$$\Rightarrow m_r \leq M_r \Rightarrow m_r \delta_r \leq M_r \delta_r \Rightarrow \sum_{r=1}^{n} m_r \delta_r \leq \sum_{r=1}^{n} M_r \delta_r .$$

Now as $n \to \infty$ every $\delta_r \to 0$ and if $\sum_{r=1}^{n} m_r \delta_r$ and $\sum_{r=1}^{n} M_r \delta_r$. approach the same limit, then f is said to be Rienann integrable or simply integrable. If f(x) is integrable in [a,b], then the limit is called the definite integral of f from a to be and its denoted by $\int_a^b f(x)dx.$

Definition :

Let f(x) be a function defined on [a,b]. If $\int f(x)dx$ = F(x), then F(b) - F(a) is called the definite integral of f(x) over [a,b]. It is denoted by $\int_a^b f(x)\ dx$. Where a is called lower limit and b is called upper limit.

Ex : $\int_a^b \sin\ x\ dx = [-\cos x]_a^b$

= (- cos b) - (- cos a)

= cos a - cos b

Ex : $\int_2^4 x^4\ dx > [\frac{x^5}{5}]_2^4$

$= \frac{4^5}{5} - \frac{2^5}{5}$

$= \frac{102}{5} - \frac{32}{5}$

$= \frac{992}{5}$

17.10 SOLVED EXAMPLES :

Examples 1 :

Evaluate $\int_0^1 (2x^6 + 4x^3 + 3)\ dx$

Solution : $\int_0^1 (2x^6 + 4x^3 + 3)\ dx$

$= \left[\frac{2x^{6+1}}{6+1} + \frac{4x^{3+1}}{3+1} + 3x\right]_0^1$

$$= \left[\frac{2x^7}{7} + \frac{4x^4}{4} + 3x\right]_0^1$$

$$= \left[\frac{2(1)^7}{7} + 1^4 + 3(1)\right] - \left[\frac{2(0)^7}{7} + 0^4 + 3(0)\right]$$

$$= \frac{2}{7} + 1 + 3$$

$$= \frac{2}{7} + 4$$

$$= \frac{30}{7}$$

Examples 2 :

Evaluate $\int_0^{\pi/2} \sin x \, dx$

Solution : $\int_0^{\pi/2} \sin x \, dx$

$$= [-\cos x]_0^{\pi/2}$$

$$= \left[-\cos\frac{\pi}{2}\right] - [-\cos 0]$$

= (-0) – (-1)

= 1

Examples 3 :

Evaluate $\int_1^2 e^{5x-4} \, dx$

Solution : $\int_1^2 e^{5x-4} \, dx$

$$= \left[\frac{e^{5x-4}}{5+1}\right]_1^2$$

$$= \left[\frac{e^{5x-4}}{6}\right]_0^1$$

$$= \frac{e^{5(1)-(4)}}{6} - \frac{e^{5(0-4)}}{6}$$

$$= \frac{e^{5-4}}{6} - e^0$$

$$= \frac{e}{6} - 0$$

Examples 4 :

Evaluate $\int_1^2 \left(\frac{x^2+2x+1}{x}\right) dx$

Solution : $\int_1^2 \left(\frac{x^2+2x+1}{x}\right) dx$

$$= \int_1^2 \left(x+2+\frac{1}{x}\right) dx$$

$$= \left[\frac{x^2}{2} + 2x + \log x\right]_1^2$$

$$= \left[\frac{2^2}{2} + 2.2 + \log 2\right] - \left[\frac{1^2}{2} + 2.1 + \log 1\right]$$

$$= \left[2+4+\log 2\right] - \left[\frac{1}{2} + 2 + 0\right] \qquad (\because \log 1 = 0)$$

$$= \frac{7}{2} + \log 2$$

Examples 5 : Evaluate $\int_2^3 \frac{2x}{1+x^2}\,dx$

Solution : Put $1 + x^2 = t$

Differentiate

$(0+2x)\,dx = dt$

$$\therefore \int \frac{2x\,dx}{1+x^2} = \left[\int \frac{dt}{t}\right]$$

$= [\log t]$

$= \log(1+x^2)$

$$\therefore \int_2^3 \frac{2x\,dx}{1+x^2} = \left[\log\left(1+x^2\right)\right]_2^3$$

$= \log(1+3^2) - \log(1+2^2)$

$= \log 10 - \log 5$

$= \log\left(\frac{10}{5}\right)$

$= \log 2$

17.11 EXERCISE :

Evaluate

1. $\int_0^1 2x^8\,dx$

2. $\int_1^2 \left(2-3x+x^2\right)dx$

3. $\int_1^2 \frac{(5x+2)^2}{3x}\,dx$

4. $\int_0^{\pi/2} \sec^2 x\,dx$

5. $\int_0^5 e^{2x+3}\,dx$

6. $\int_3^4 \frac{1}{x^3}\,dx$

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Lesson - 18

LINEAR PROGRAMMING

OBJECTIVES:

By the study of this lesson, you will be able to understand the meaning, definition and scope and significance of Linear programming function with examples.

STRUCTURE:

18.1 Introduction

18.2 Definitions of Linear programming

18.3 Mathematical formulation of Linear programming problems.

18.4 Summary

18.5 Questions

18.6 Exercises.

18.1 INTRODUCTION :

Today, the business manager face problems whcih involves decisioin making regarding allocation of limited resources to various activities with objective of yielding maximum production or to minimise the cost of production or maximum profit. These decision problems can be formulated and solved as mathematical programming problems.

Linear programming is an optimisation technique. It is a technique for specifying how to use limited resources of a business to obtain a particular objective i.e, least cost, maximum production and profit etc. The term " Linear " indicates that all realationships involved in the system are linear. " programming " is another word normally determining a particular plan of action amongst several available alternatives.

We will explain an example of linear programming problem and define " Linear programming".

Examples :

A micro computer manufacturer makes two models, the x_1 and x_2. The profits on each $x_1$and x_2 are Rs. 400 and Rs. 500 respectively. The manufacture wishes to determine the quantity of each model to produce in order to maximise his total yearly profit.

In each year his supply of components is enough to make up to a total of 600 computers. The x_2 requires twice as much machine capacity as the x_1, and there is enough machine to produce up to 700 computers per year. The x_2 also requires four times as much labour as the x_1, and there are enough man - hours available each year to produce upto 1,200 computers.

The manufacturer's problem is to decide how many of each model to produce in order to maximise his total profit, but obviously he is constrained by the resources available each year. His problem may be formluated mathematically as follows :

Objective.

Maximise 400 x_1 + 500 x_2

Where x_1 = number of x_1 produced

x_2 = number of x_2 produced.

Constraintis

$x_1 + x_2 \leq 600$ (Supply constraint)

$x_1 + 2x_2 \leq 700$ (machine capacity constraint)

$x_1 + 4x_2 \leq 1200$ (labour constraint)

$x_1 , x_2 \geq 0$

			Total
Supply	1	1	600
Machine capacity	1	2	700
Labour	1	4	1200
Profit Maximissation	400	500	

The first constraint is obvious. The second constraint can be worked out as follows. Suppose each x1 requires y units of machine capacity. Then each x2 will require 2y units of machine capacity, and the total number of units of machine capacity available will be 700y.

The third constraint can be obtained in the same way.

In standard notation this is written as

maximise z = 400x1 + 500 x2

Subject to

$x_1 + x_2 \leq 600$

$x_1 + 2x_2 \leq 700$

$x_1 + 4x_2 \leq 1200$

$x_1 , x_2 \geq 0$

The unknown x_1 and x_2 whose optimal values are to be determined are called the "decision variables".

The function $z = 400x_1 + 500x_2$

is called the " objective function", as it represents the criterion for selecting the optimal values of the decision variables x_1 and x_2 to achieve the objective.

The constraints imposed by the resoures available, i.e

$$x_1 + x_2 \leq 600$$

$$x_1 + 2x_2 \leq 700$$

$$x_1 + 4x_2 \leq 1200$$

are called the structural constraints, as they directly affect the possible combinations of values for the decision variables.

The last constraint $x_1 , x_2 \geq 0$

is the "non negativity " constraint. Which stipulates that the values of the decision variables must be positive or at least zero.

A set of values for the decision variables which satisfy both the structural and nonnegativity constraints is a feasible solution to the problem. A feasible solution which achieves the objective is an "optimal solution".

18.2 DEFINITION OF LINEAR PROGRAMMING :

Having seen an example of a linear programming problem and its mathematical formulation. We now give a definition of linear programming.

Linear programming defines a class of problems with the following characteristics.

1) All the decision variables are nonnegative.
2) The objective functions is a linear function of the decision variables and the objective is either maximisation or minimisation.
3) The structural constraints are linear inequalities (or eqations) in the decision variables.

It is easy to verify that above example is indeed a linear programming problem.

18.3 MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING PROBLEMS :

Given a written description of a linear programming problem. We have to formulate the problem mathematically before we can attempt to solve it. There are four basic steps in deriving its mathematical formulation :

1. Identify the decision variables.

2. Identify the objective function to be optimised (maximised or minimised) and express it as a linear function of the decision variables.

3. Identify all the structural constraints in the problem, and express them as linear inequalities (or equations) in the decision variables.

4. Specify the nonnegativity constraint.

The objective function, the set of constraints and the non negativite constraint together form a linear programming problem.

Example 1 :

A manufacturer produces two types of models M_1 and M_2 each M_1 model requires 4 hours of grinding and 2 hours of polishing. Where as each M_2 model requires 2 hours of grinding and 5 hours of polishing. The manufacture has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is Rs. 3.00 and on an M_2 model is Rs. 4.00 whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models so that he may make the maximum profit in a week ?

Solution :

Decision variables : Let

x_1 = number of units of M_1 model, and

x_2 = number of units of M_2 model,

Objective function : The object of the manufacturer is to determine, the number of M_1 and M_2 models so as to maximize the total profit viz. $z = 3x_1 + 4x_2$.

Constraints : For grinding since each M_1 model requires 4hours and each M_2 model requires 2 hours the total number of grinding hours needed per week is given by $4x_1 + 2x_2$

Similarly for polishing, the total number of polishing hours needed per week is $2x_1 + 5x_2$

Further since the manufacturer does not have more than hours of grinding x 40 hours = 80hours of grinding and 3hours of polishing x 60 hours = 180 hours of polishing, the time constraints are

$$4x_1 + 2x_2 \leq 80$$

$$2x_1 + 5x_2 \leq 180$$

Non negativity constraints: Since the production of negative number of models is meaningless, we must have $x_1 \geq 0$ and $x_2 \geq 0$

Hence, the manufacturer's allocation problem can be put in the following mathematical form.

Objective function

Maximise z = $3x_1 + 4x_2$.

Subject to

$4x_1 + 2x_2 \leq 80$

$2x_1 + 5x_2 \leq 180$

$x_1, x_2 \geq 0$

Example 2 :

The manager of an oil refinery must decide on the optimum mix of 2 possible blending processes of which the inputs and outputs production run are as follows.

	Input		Output	
Process	**crude A**	**curdeB**	**Gasoline x**	**Gasoline Y**
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crudes A and B 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline x and 130 units of gasoline y must be produced. The profits per production run from process 1 and process 2 are Rs. 4 and Rs.5 respectively. Formulate the problem for maximising the profit.

Solution :

Decision variables : Let

x_1 = number of units of Gasoline from process 1

x_2 = number of units of Gasoline from process 1

Objective function : Maximize z = $4x_1 + 5x_2$.

Constraints :

$6x_1 + 5x_2 \leq 250$, $4x_1 + 6x_2 \leq 200$ (Crude A,B) ,

$6x_1 + 5x_2 \geq 150$, $9x_1 + 5x_2 \geq 130$ (Gasoline x,y)

$x_1 \geq 0$ and $x_2 \geq 0$

Example 3 :

The owner of Metro sports wishes to determine how many advertisements to place in the selected three monthly magazines. A, B and C. His objective is to advertise in such a way that total exposure to principal buyers of expensive sports good is maximized. Percentages of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used :

	Magazine A	Magazine B	Magazine C
Readers	1lakh	0.6 lakh	0.4lakh
Principal Buyers	10%	15%	7%
Cost per advertisement (Rs)	5000	4500	4250

The budgeted amount is at most Rs. 1 lakh for the advertisements. The owner has already decided that magazine A should have no more than 6 advertisements and that B and C each have at least two advertisements. Formulate on LP model for the problem.

Solution :

Decision variables : Let

x_1 = number of insertions in magazine A

x_2 = number of insertions in magazine B and

x_3 = number of insertions in magazine C

Objective function : Maximize

$$z = 0.10 \text{ x } 1,00,000\ x_1 + 0.15 \text{ x } 60,000\ x_2 + 0.7 \text{ x } 40,000\ x_3$$

$$= 10,000\ x_1 + 9,000\ x_2 + 2,800\ x_3$$

Constraints :

$5,000\ x_1 + 4,500\ x_2 + 4250\ x_3 \leq 1,00,000$ (budgeting) ,

$x_1 \geq 6$, $x_2 \geq 2$, $x_3 \geq 2$ (advertising)

$x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$

Example 4 :

Three grades of coal, A,B and C contain ash and phosporus as impurities. In a particular industrial process a fuel obtained by blending the above grades. Containing not more than 25% ash and 0.03% phosphorus is required. The maximum demand of the fuel is 100 tons. percentage impurities and costs of the various grades of coal are shown below. Assuming that there is an unlimited supply of each grade of coal and there is no loss in blending, formulate the blending problem to minimise the cost.

Coal Grade	% ash	% Phosphours	Cost per ton in Rs.
A	30	0.20	240
B	20	0.04	300
C	35	0.03	280

Solution :

Decision variables : Let

x_1 = tons of grade A coal

x_2 = tons of grade B coal, and

x_3 = tons of grade Coal

Objective function :

$$\text{Maximize } z = 240\ x_1 + 300\ x_2 + 280\ x_3$$

Constraints :

$0.3\ x_1 + 0.2\ x_2 + 0.35\ x_3 \leq 0.25$ (ash) ,

$\frac{0.20}{100} x_1 + \frac{0.04}{100} x_2 + \frac{0.03}{100} x_3 \leq \frac{0.03}{100}$ (phosphorus) ,

$x_1 + x_2 + x_3 \leq 100$ (demand of fuel)

$x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$

Example 5 :

A ship has three cargo holds, forward, aft and centre, the capacity limits are :

Forward	2,000 tonnes	100,000 m^3
Centre	3,000 tonnes	135,000 m^3
After	1,500 tonnes	30,000 m^3

The following cargoes are offered, the ship owners may accept all or any part of each commodity.

Commodity	**Amount tonnes**	**Volume (tonne) m3**	**Profit per tonne (Rs)**
A	6,000	60	60
B	4,000	50	80
C	2,000	25	50

In order to preserve the trim of the ship, the weight in each hold must be proportional to the capacity in tonnes. The objective is to maximize the profit. Formulate the linear programming model for this problem.

Solution :

Decision variables : Let x_{1A} , x_{2A} , x_{3A} be the weights (in Kg) of the commodity A to be accommodated in forward, centre aft portions respectively. Similarly, let x_{1B} , x_{2B} , x_{3B} and x_{1C} , x_{2C} , x_{3C} be the corresponding weights (in kg) of B and C.

Objective function : Maximize

$$z = 60\,(x_{1A}\ , x_{2A}\ , x_{3A}) + 80\,(x_{1B}\ , x_{2B}\ , x_{3B}) + 50\,(x_{1C}\ , x_{2C}\ , x_{3C})$$

Constraints :

$x_{1A}\ , x_{2A}\ , x_{3A} \leq 6,000$

$x_{1B}\ , x_{2B}\ , x_{3B} \leq 1,000$ (commodity cargo)

$x_{1C}\ , x_{2C}\ , x_{3C} \leq 2,000$

$x_{1A}\ , x_{1B}\ , x_{1C} \leq 2,000$

$x_{2A}\ , x_{2B}\ , x_{2C} \leq 3,000$ (Weight capacity)

$x_{3A}\ , x_{3B}\ , x_{3C} \leq 1,500$

$60x_{1A}\ , x_{1B}\ , x_{1C} \leq 1,00,000$

$60x_{2A}\ , x_{2B}\ , x_{2C} \leq 1,35,000$ (volume capacity)

$60x_{3A}\ , x_{3B}\ , x_{3C} \leq 30,000$

Example 6 :

A contractor has been assigned the excavation work of a canal and of the head works on a project. In order to ensure a balanced progress on the entire work the management has imposed certain conditions on his working. The excavation of the canal is more profitable than that of the head works, but he has to abide by the conditions of contract. In addition, he has his own limitations on man power and equipment. It is desired to find the optimum amount of excavation on the two works which he should undertake so that his profits are maximized and he satisfies the constraints of the contract and of the resources. These constraints are given below :

i. The difference in the quantity of earth work done on the two works does not exceed 2 units in a day.

ii. The difference between the quantity of canal excavation and of twice the head works evacuation does not exceed 1 unit in a day.

iii. Each unit of canal excavation done in a day requires one unit of man power and one unit of machines, and each unit of head works excavation requires 2 units of man power and one unit of machines. Maximum available man power is 10 units and machines 6 units.

The contractor stipulates a profit of 2 units for each unit of canal excavation and half of this for each unit of head works excavation

Solution :

Decision variables : Let the quantity of each excavation that the contractor should do every day. i.e x_1 units and quantity of head works excavation x_2 units. It is obvious that the variables x_1 and x_2 are non - negative.

The objective function is Max $z = 2x_1+x_2$ (maximise) and the constraint relationships are

$x_1 + 2x_2 \leq 10$ (man power constraint)

$x_1 + x_2 \leq 6$ (machine constraint)

$x_1 - x_2 \leq 2$ (contract constraint)

$x_1 - 2x_2 \leq 1$

$x_1 , x_2 \geq 0$ (contract constraint)

Example 7 :

Rolls of paper having a fixed length and width of 180 cm are being manufactured by a paper mill. These rolls have to be cut to satisfy the following demand.

Width	80cm	45cm	27cm
No.of Rolls :	200	120	130

Obtain the linear programming formulation of the problem to determine the cutting pattern, so that the demand is satisfied and wastage of paper is a minimum.

Solution :

Various alternatives for the number of rolls are given below :

Feasible Pattern of cutting	**No. of rolls cut**	**Wastage per roll (cm)**	**Rolls obtained from each mother roll of width**		
			80cm	**45cm**	**27cm**
80+80	x_1	20	2	-	-
80+45+45	x_2	10	1	2	-
80+45+27+27	x_3	1	1	1	2
80+27+27+27	x_4	19	1	-	3
45+45+45 +45	x_5	0	-	4	-
45+45+45+27	x_6	18	-	3	1
45+45++27+27+27	x_7	9	-	2	3
45+27+27+27+27+27	x_8	0	-	1	5
27+27+27+27+27+27	x_9	18	-	-	6

Thus the linear programming problem is :

Minimize $z = 20\ x_1 + 10\ x_2 + x_3 + 19\ x_4 + 18x_6 + 9\ x_7 + 18\ x_9$

Subject to the constraints :

$2\ x_1 + x_2 + x_3 + x_4 = 200$ (80 cm rolls)

$2\ x_2 + x_3 + 4\ x_5 + 3\ x_6 + 2\ x_7 + x_8 = 120$ (45 cm rolls)

$2\ x_3 + 3\ x_4 + x_6 + 3\ x_7 + 5\ x_8 + 6\ x_9 = 130$ (27 cm rolls)

$x_j \geq 0 = j = 1,2,3,..........9$

18.4 SUMMARY :

Linear programming problem aims at maximising production or minimising the cost of production or maximising the profit. These problems are formulated and solved as mathematical programming problems.

18.5 QUESTIONS :

1. What do you understand by Linear programming problem ?
2. Explain importance of Linear programming problem.
3. Define the Linear programming problem.
4. Explain the mathematical formulation of Linear programming with examples.

18.6 EXERCISES :

1. At the mancunian water works, two chemicals are added to the water. Chemical A is added to soften the water, and chemical B is a purifier used for health protection purposes. To maintain minimum standards of water softness and health protection, at least 150 Ibs. of A and 100 Ibs. of B must be added daily.

   There are only two products available which contain a mixture of the two chemicals. Water soft contains 8 Ibs. of A and 3 Ibs. of B and costs 80P per unit, water pure contains 4 Ibs. of A and 9 Ibs. of B and cost Rs. 1 per unit.

   Formulate this as a linear programming problem to minimize the daily softening and purifying cost.

2. The shooting star company manufactures three kinds of foot ball machines each requiring a different manufacturing technique. The man united requires 15 hours of labour, 5 hours of testing and yields profit of Rs. 300. The man city requires 10 hours of labour, 4 hours of testing and fetches a profit of Rs. 200. The Robson's choice requires 5 hours of labour, 3hours of testing and yields a profit of Rs. 100.

   There are 2,000 hours of labour and 1,000 hour of testing available according to the company's sales forecast, the demands for the there models are no more than 50,100, 150 resepectively for the man united, the man city and the Robson's choice.

The company wishes to determine the optimal production plan that will maximise its total profit. Formulate its problem as a linear programming problem.

3. A cosmetics manufacturer wishes to plan an advertising compagian in three media, magazines, radio, and television. The purpose of compaign is to reach as many potential female customers as possible. A market study gives the following information about the potential of an advertising unit in each of the media (the numbers are in 10,000's).

	Magazines	**Radio**	**Television**
total audience reached	1	4	7
Potential female Customers reached	1	4	2

The manufacturer does not want to buy more than 3 advertising units. It also requiers that the advertisement do not reach an audience of more than 90,000.

Formulate the manufacturer's compaign as a linear programming problem.

4. A firm manufacturers headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and I grain of codeine. Size B contains 1 grain of aspirin. 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard LPP.

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Lesson - 19

LINEAR PROGRAMMING - GRAPHICAL METHOD

OBJECTIVES:

By the study of this chapter, you will be able to understand in detail the Graphical Method of solving Linear Programming Function with examples.

STRUCTURE:

19.1 GRAPHIC METHOD - INTRODUCTION :

The graphic solution is used in solving two variables Linear programming problems.

19.2 EXAMPLES :

Example 1 :

Let us consider the problem of

Maximise Z = 10 x_1 + 20 x_2

Subject to :

$5x_1 + 3x_2 \leq 30$ ....(i)

$3x_1 + 6x_2 \leq 36$ ....(ii)

$2x_1 + 5x_2 \leq 20$ ....(iii)

where $x_1 , x_2 \geq 0$ ....(iv)

Solution :

Convert the inequality equation to equality equation

$5x_1 + 3x_2 = 30$

$3x_1 + 6x_2 = 36$

x_1	0	10
x_2	4	0

$2x_1 + 5x_2 = 20$

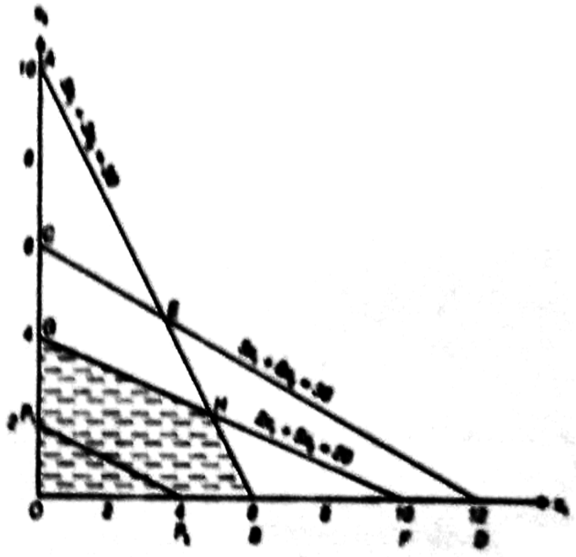
Finding points for the above equations , we get

$5x_1 + 3x_2 = 30$

Given $x_1 = 0$

x_1	0	6
x_2	10	0

We get $5(0) + 3x_2 = 30$



$3x_2 = 30$

$x_2 = \frac{30}{3} = 10$

Given $x_1 = 6$

We get $5(6) + 3x_2 = 30$

$30 + 3x_2 = 30$

$3x_2 = 30 - 30$

$x_2 = \frac{0}{3} = 0$

Taking equation

$3x_1 + 6x_2 = 36$

Given the value for $x_1 = 0$

x_1	0	12
x_2	6	0

We get $3(0) + 6x_2 = 36$

$6x_2 = 36$

$x_2 = \frac{36}{6} = 6$

Given the value for $x_1 = 12$

We get $3x_1 + 6x_2 = 36$

$3(12) + 6x_2 = 36$

$36 + 6x_2 = 36$

$6x_2 = 0$

$x_2 = 0$

Taking equation

$2x_1 + 5x_2 = 20$

Given the value for $x_1 = 0$

We get $2(0) + 5x_2 = 20$

$5x_2 = 20$

$x_2 = 4$

Given the value for $x_1 = 10$

We get $2(10) + 5x_2 = 20$

$20 + 5x_2 = 20$

$5x_2 = 0$

$x_2 = 0$

AB repersents $5x_1 + 3x_2 = 30$

CD repersents $3x_1 + 6x_2 = 36$

FG repersents $2x_1 + 5x_2 = 20$

It can be seen that the region OGHBO satisfies the constraints i, ii, iii, iv. This shaded area is called as the feasibility region. Hence, the feasibility region is one for which all points within and on its boundary satisfy all the constraints of the given problem. The region OGHBO is also called as solution space. It is interesting to note that the straight line CD does not form the boundary constraint (ii) is called as a redundant constraint, in the sense that it does not effect the solution.

Now it is possible to choose any point in the solution space (feasibility region) which would maximise the objective function.

All values of x_1 and x_2 falling in the feasibility region OGHBO satisfy the given set of constraints.

We have $Z = 10\ x_1 + 20\ x_2$

The coordinates at 0 are $x_1 = 0 \quad x_2 = 0$

G are $x_1 = 0 \quad x_2 = 4$

H are $x_1 = 4.74$, $x_2 = 2.11$

B are $x_1 = 6$, $x_2 = 0$

Z at 0 = 0

Z at G = 80

Z at H = 47.37 + 42.11 = 89.48

Z at B = 60

Max occurs at H and Z = 89.48

Example 2 :

Solve graphically the follwoing LPP

Maximise $Z = 3\ x_1 + 2\ x_2$

Subject to :

$-2x_1 + x_2 \leq 1$

$x_2 \leq 2$

$x_1 + x_2 \leq 3$

where $x_1 \geq 0$, $x_2 \geq 0$

Solution :

Convert the inequality equations to equality equations

$-2x_1 + x_2 = 1$

$x_2 = 2$

$x_1 + x_2 = 3$

Finding points for the above equations we get

$-2x_1 + x_2 = 1$

Given $x_1 = 0$

x_1	0	-1/2
x_2	1	0

We get $-2(0) + x_2 = 1$

$x_2 = 1$

Given $x_1 = -1/2$

We get $-2(-1/2) + x_2 = 1$

$1 + x_2 = 1$

$x_2 = 0$

Taking equation

$x_1 + x_2 = 3$

Given $x_1 = 0$

x_1	0	3
x_2	3	0

We get $0 + x_2 = 3$

$x_2 = 3$

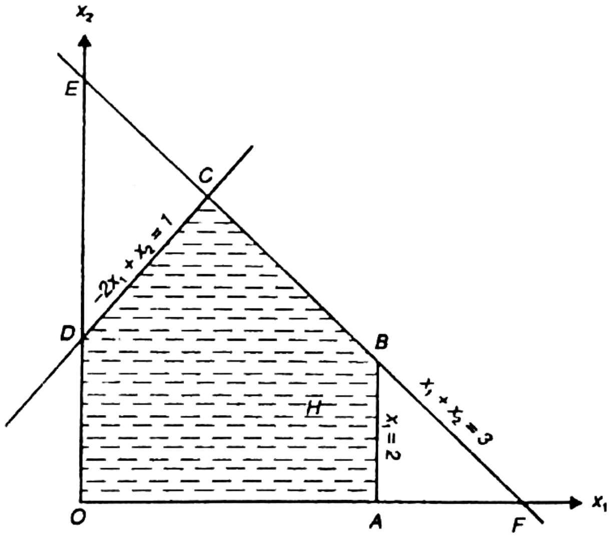
When $x_1 = 3$

We get $3 + x_2 = 3$

$x_2 = 3 - 3 = 0$

DC represents $-2x_1 + x_2 = 1$

EF represents $x_1 + x_2 = 3$



The feasible region satisfying all the requried constraints is ODCBAO

We have $Z = 3\,x_1 + 2\,x_2$

Co-ordinates at O are (0,0) $\therefore Z = 0$

Co-ordinates at D are (0,1) $\therefore Z = 2$

Co-ordinates at C are (2/3 , 7/3) $\therefore Z = 6.7$

Co-ordinates at B are (2,1) $\therefore Z = 8$

Co-ordinates at A are (2,0) $\therefore Z = 6$

Maximum is 8 at $x_1 = 2$, $x_2 = 1$

Example 3 :

Solve graphically

Minimum $Z = 4\,x_1 + 2\,x_2$

Subject to :

$x_1 + 2x_2 \geq 2$

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

where $x_1 \geq 0$, $x_2 \geq 0$

Solution :

Convert the inequality equation to equality equation

$x_1 + 2x_2 = 2$

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 = 6$

Finding points for the above equation we get

$x_1 + 2x_2 = 2$

Given $x_1 = 0$

x_1	0	2
x_2	1	0

We get $0 + 2x_2 = 2$

$$x_2 = \frac{2}{2} = 1$$

Given $x_1 = 2$

We get $2 + 2x_2 = 2$

$2x_2 = 0$

$x_2 = 0$

Taking equation

$3x_1 + x_2 = 3$

Given $x_1 = 0$

x_1	0	1
x_2	3	0

We get $3(0) + x_2 = 3$

$x_2 = 3$

Given $x_1 = 1$

We get $3(1) + x_2 = 3$

$x_2 = 0$

Taking equation

$4x_1 + 3x_2 = 6$

Given $x_1 = 0$ we get

$4(0) + 3x_2 = 6$, $x_2 = 2$

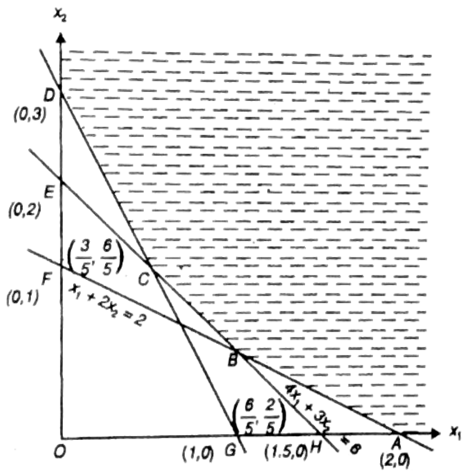
Given $x_1 = 1.5$

x_1	0	1.5
x_2	2	0

We get $4(1.5) + 3x_2 = 6$

$3x_2 = 0$

$x_2 = 0$



In the above figure.

AF represents $x_1 + 2x_2 = 2$

DG represents $3x_1 + x_2 = 3$

EH represents $4x_1 + 3x_2 = 6$

The feasible region satisfying all the requried constraints is ABCD

$\therefore$ $Z = 4x_1 + 2x_2$

Co ordinates at A are (2,0) $\therefore Z = 8$

Co ordinates at Bare (1.2,0.4) $\therefore Z = 5.60$

Co ordinates at C are (0.60 , 1.2) $\therefore Z = 4.80$

Co ordinates at D are (0,3) $\therefore Z = 6.00$

$\therefore$ Minimum is 4.80 at C, $x_1 = 0.60$, $x_2 = 1.2$

19.3 SUMMARY :

Graphical method of Linear programming problem deals with solving the problem of Maximising the production minimising cost of production and maximising the profit.

19.4 QUESTIONS & EXERCISES :

1. Explain the Graphical method of solving Linear Programming problem.
2. Solve the following Linear programming problems graphically and shade the region representing the feasibility region and feasible solutions.

Min $Z = 5x_1 - 2x_2$

Subject to :

$2x_1 + 3x_2 \geq 1$

$x_1 \geq 0$ and $x_2 \geq 0$

[Ans : $x_1 = 0$, $x_2 = 1/3$, Min $Z = -2/3$]

3. Max $Z = 5x_1 + 3x_2$

Subject to :

$3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1 \geq 0$ and $x_2 \geq 0$

[Ans : $x_1 = 20/19$, $x_2 = 45/19$, Max $Z = 235/19$]

4. Max $Z = 2x_1 - 3x_2$

Subject to :

$x_1 + x_2 \leq 1$

$3x_1 + x_2 \leq 4$

$x_1 \geq 0$ and $x_2 \geq 0$

[Ans : $x_1 = 20/19$, $x_2 = 45/19$, Max $Z = 3$]

5. Max $Z = 6x_1 + 4x_2$

Subject to :

$-2x_1 + x_2 \leq 2$

$x_1 - x_2 \leq 2$

$3x_1 + 2x_2 \leq 9$

$x_1 \geq 0$ and $x_2 \geq 0$

[Ans : $x_1 = 13/5$, $x_2 = 3/5$, Max $Z = 8$]

6. $2x_1 + 3x_2 \leq 6$

$x_1 + x_2 \leq 4$

$x_1 , x_2 \geq 0$ Max $Z = x_1 + 15x_2$

[Ans : $x_1 = 12/5$, $x_2 = 2/5$, Max $Z = 42/5$]

7. $-2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1 + x_2 \leq 3$

$x_1 \geq 0$ and $x_2 \geq 0$ Max $Z = x_1 + x_2$

[Ans : Infinite number of maximal solutions]

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Lesson - 20

LINEAR PROGRAMMING - SIMPLEX METHOD

OBJECTIVES:

By the study of this Lesson you will be able to understand the basic terms involved and method of solving the Liner programming problem under simplex method with examples.

STRUCTURE:

20.1 INTRODUCTION :

The Linear programming problems with more than three decision variables, the graphical method becomes impossible. So we turn our attention to an algebraic method which can deal with any number of decision variables namely the simplex method. The simplex method is the name given to the solution, algorithm for solving LP problems developed by George B. Dantzig in 1947.

20.2 BASIC TERMS INVOLVED IN SIMPLEX PROCEDURE :

We shall introduce certain terms which are relevant for solving a linear programming problem through simplex procedure.

1. **Standard Form :** A linear programme in which all the constraints are written as equalities. The optimum solution of the standard form of a linear programme is the same as the optimum solution of the original formulation of the linear programme.
2. **Slack Variable :** A variable added to the "left - hand side of is less than or equal" to constraint to convert the constraint into an equality. The value of this variable can usually be interpreted as the amount of unused resource.
3. **Surplus Variable :** A Variable subtracted from left - hand side of a greater than or equal to constraint to convert the constraint into an equality. The value of this variable can usually be interpreted as the amount over and above the required minimum level.
4. **Basic solution :** For a general linear programme with 'n' variable and 'm' constraints a basic solution may be found by setting (n-m) of the variables equal to zero and solving the constraint equations for the values of the other 'm' variables. If a unique solution exists, it is a basic solution.

5. **Basic feasible solution :** A basic solution which is also in the feasible region (i.e. it satisfies the non - negativity requirement), A basic feasible solution corresponds to a corner point of the feasible region.

6. **Tableau form :** The form in which a linear programme must be written prior to setting up the initial simplex table. When a linear programming is written in this form, its matrix contains 'm' unit columns corresponding to basic variables and the values of these basic variables given by the quantity column. A further requirement is that the entries in the quantity column be greater than or equal to zero. The requirement provides us with a basic feasible solution.

7. **Simplex Table :** A table used to keep track of the calculations made at each iteration when the simplex solution method is employed.

8. **Product Mix :** A column in the simplex table that contains all of the variables in the solution.

9. **Basis :** The set of variables which are not restricted to equal zero in the current basic solution and are listed in the product mix column. The variables which make up the basis are termed basic variables and the remaining variables are called non - basic variables.

10. **Iteration :** An iteration of the simplex method consists of the sequence of steps performed in moving from one basic feasible solution to another.

20.3 EXAMPLES :

Example 1 :

A furniture manufacturer produces & sells desks, chairs and book shelves. He has no difficulty in selling the items. However, limited availability of machine time, labour & floor space restricts production. Data on usage of resources, supplies & profits on items are given below.

Particulars	**Desk**	**Chair**	**Book Shelf**	**Supply**
Machine hour/ unit	8	4	5	1000 hrs.
Labour hour/ unit	5	3	3	650 hrs.
Florr space Sq.ft/unit	9	6	9	1260 hrs.
Constribution (Rs) p.u	270	144	225	

1. Formulate the above as a LPP
2. Use simplex method to find the opitimal solution. What is the maximum total profit ?
3. The personal manager of the co, claims that by recruiting additional labour force, the profits can be increased. Is the claim valid ? Explain with reasons.

Solution : LP formulation

Let x_1 x_2 and x_3 numbers of the desk, chairs & book shelves be respectively manufactured. The objective function is

Max $Z = 270\ x_1 + 144\ x_2 + 225\ x_3$

Subject to :

1. $8\ x_1 + 4\ x_2 + 5\ x_3 \leq 100$ (Constraint on mach. hrs)
2. $5\ x_1 + 3\ x_2 + 3\ x_3 \leq 650$ (Constraint on labour. hrs)
3. $9\ x_1 + 6\ x_2 + 9\ x_3 \leq 1260$ (Constraint on floor. hrs)

   (x_1 , x_2 , $x_3 \geq 0$)

Introducing slack variables S_1, S_2 and S_3 respectively in constrainst 1,2, and 3. The above LPP is modified as

Max $Z = 270\ x_1 + 144\ x_2 + 225\ x_3 + OS_1 + OS_2 + OS_3$

1. $8\ x_1 + 4\ x_2 + 5\ x_3 + S_1 = 100$
2. $5\ x_1 + 3\ x_2 + 3\ x_3 + S_2 = 650$
3. $9\ x_1 + 6\ x_2 + 9\ x_3 + S_3 = 1260$

   (x_1 , x_2 , x_3 , S_1, S_2 , $S_3 \geq 0$)

Number of variables = 6,

No. of equations = 3.

6-3 = 3 variables are equal to 0.

x_1 , x_2 & x_3 = 0 and $S_1 = 1000$; $S_2 = 650$ and $S_3 = 1260$

Simplex Table

FR - Fixed ratio

Prog - Programme

R.R - Replacement Ratio

N.E.R - Net Evaluation Row

K.C - Key column

K.R - Key Row

K.E - Key Element

N.K.R - Non - Key Row

Pft.P.U. - Profit Per Unit

Intial Simplex table

FR	Prog	Pft P.U.	Qty	210 x_1	144 x_2	225 x_3	0 S_1	0 S_2	0 S_3	RR
	S_1	0	1000	8 K.E	4	5	1	0	0	1000/8 = 125 (KR)
5/8	S_2	0	650	5	3	3	0	1	0	650/5 =130
9/8	S_3	0	1260	9	6	9	0	0	1	1260/9 =140
	NER			270 K.C	144	225	0	0	0	

Iteration 1

FR	Prog	Pft P.U.	Qty	x_1	x_2	x_3	S_1	S_2	S_3	RR
5/27	x_1	270	125	1	1/2	5/8	1/8	0	0	125x 8/3 = 200
1/27	S_2	0	25	0	1/2	-1/8	-5/8	1	0	-ve
	S_3	0	135	0	3/2	27/8 K.E.	-9/8	0	1	135 x 8/27 = 40 (K.R.)
	NER			0	9	225- /4 K.C.	-ve	0	0	

Iteration 2

FR	Prog	Pft P.U.	Qty	x_1	x_2	x_3	S_1	S_2	S_3	RR
	x_1	270	100	1	2/9	0	1/3	0	-5/27	
	S_2	0	30	0	5/9	0	-2/3	1	1/27	
	x_3	225	40	0	4/9	1	-1/3	0	8/27	
	NER			0	-ve	0	-ve	0	-ve	

Hence the optimal solution is given by

x_1 = 100 x_2 = 0 , x_3 = 40 and S_1= 0 ; S_2 = 30 , S_3 = 0

Max Z = [270 x 100 + 225 + 40] = [27000 + 9000] = Rs. 36000

Hence the co, must produce 100 numbers of desks 40 numbers of book shelves in order to derive the max. total profit of Rs. 36,000, x_2 = 0 Hence the co. must stop the production of chairs.

At the optimal stage the slack variable S_2 = 30. It means that the resources namely labour in whose constraints the slack variables has been introduced has got ideal capacity of 30 hrs. Hence, the claim of the personal manager is totally invalid.

Steps used for calculation

Initial simplex Table :

1. First enter all the values in the table
2. In programe columns enter S_1 ;S_2 ; S_3
3. In profit per unit column enter co- efficients of S_1 ;S_2 ; S_3 in objective functions i.e OS_1; OS_2 ; OS_3 i.e., O.
4. In quantity column enter values written on right hand side of constraints equations i.e, 100, 650 and 1260.
5. In x_1 column enter coefficients of x_1 of all three constraints i.e., 8,5,9.
6. In x_2 column enter coefficients of x_2 of all three constraints i.e., 5,3,3
7. In x_3 column enter coefficients of x_3 of all three constraints i.e., 9,6,9
8. In S_1 ;S_2 & S_3 column enter unit matrix, if only S_1 & S_2 is there enter 2x2 unit matrix.

9. Enter NER (Net Evaluation Row) i.e, coefficients of x_1 , x_2, x_3 , S_1, S_2 and S_3 of objective function i.e , 270, 144, 225, 0,0,0.

10. Take highest +ve value in NER row i.e., 270 and denote as key column.

11. Find values in R.R column i.e.

$$R.R = \frac{\text{Element in Qty column}}{\text{Element in key column}}$$

i.e., $\frac{1000}{8}$ =125 , $\frac{650}{5}$ =130 , $\frac{1260}{9}$ =140

12. In R.R column select the least positive value and denote as key row (K.R)

13. The intersection value of K R and KC is key element (K.E)

14. Now find FR column key element row is nil in FR column

$$\frac{\text{Element in the K.C}}{\text{Key element}}$$

i.e., $\frac{5}{8}, \frac{9}{8}$

Iteration 1

1. Introduce x_1 and drop S_1 (This is done taking KE column)
2. In programme column enter x_1 S_2 , S_3 .
3. In profit per unit column enter coefficients of x_1 S_2 , S_3 of objective function i.e, 270,0,0.
4. Convert the key element to, i.e by dividing through out the row by 8 we get 125, 1,

   $\frac{4}{8}\left(\frac{1}{2}\right), \frac{5}{8}, \frac{1}{8}$, 0,0.

5. Finding elements of non - key elements.

   New element

   = Existing element - (FR x corresponding element in the key row)

   = 650 - (5/8 x 1000) = 25

   = 5.(518 x8) = 0

   = 3- (5/8 x 4) = $\frac{6-5}{2} = \frac{1}{2}$

$= 3. (518 \times 5) = \frac{24 - 25}{8} = \frac{1}{8}$

= 0 - (5/8 x 1) = -5/8

1- (5/8 x 0) = 1,

0 - (5/8 x 0) = 0

= 1260 - (9/8 x 1000) = 135

$= 6 - (9/8 \times 4) = \frac{25 - 9}{2} = \frac{3}{2}$

$= 9 - (9/8 - 5) = \frac{72 - 45}{8} = \frac{27}{8} = 0 - (9/8 + 1) = - 9/8$

= 0 - (9/8 x 0) = 0

= 1 - (9/8 x 0) = 1

6. NER of x_1 is Zero (0) because it is a key column element

7. NER of x_2 = Exisiting element - (KC element of initial simplex Tablex Key Row element of x_2 in Iteration 1)

   $= 144 - (270 \times \frac{1}{2}) = 9$

   NER of $x_3 = 225 - \left(270 \times \frac{1}{8} \right) = \frac{225}{4}$

   NER of $S_1 = 0 - \left(270 \times \frac{5}{8} \right) = - 33.6$ (-ve)

   NER of S_2 = 0 - (270 x 0) = 0

   NER of S_3 = 0 - (270 x 0) = 0

8. Since in NER row the have positive values it has not rached optiomal stage.

9. Now repeat same procedure i.e, consider highest positive value for NER i.e, as $\frac{225}{4}$ as K.C

10. Find R.R = $\frac{\text{Element in Qty column}}{\text{Element in key column}}$

$= \frac{125}{5/8} = 225 \text{ x } \frac{8}{5} = 200$

$= \frac{25}{-1/8} = 25 - 8 = -200 \text{ (-ve)}$

$= \frac{135}{27/8} = 135 \text{ x } \frac{8}{27} = 40$

Among these values least positive value is 40 so it is KR

11. The intersection of KR and KC is KE.

12. Find FR i.e = $\frac{\text{Element in the Key column}}{\text{Key element}}$

$= \frac{5/8}{27/8} = \frac{5}{8} \text{x} \frac{8}{27} = \frac{5}{27}$

FR = $\frac{-1/8}{27/8} = \frac{-1}{8} \text{x} \frac{8}{27} = \frac{1}{27}$

FR of KE row is nil

Iteration 2

1. Introduce x_3 and drop S_3 (Considering KE column)
2. In programme column enter x_1 S_2 , x_3 .
3. In profit per unit column enter coefficients of x_1 S_2 , x_3 of objective function i.e, 270,0,225.
4. Convert the key element to 1 i.e by dividing through out the row by 27/8 we get

$135\text{x} \frac{8}{27} = 40$, $0 \text{ x } \frac{8}{27} = 0$

$\frac{3}{2} \text{x} \frac{8}{27} = \frac{4}{9}$

$\frac{27}{8} \text{x} \frac{8}{27} = 1$

$$\frac{-9}{8} \times \frac{8}{27} = \frac{1}{3}$$

$$1 \times \frac{8}{27} = \frac{8}{27}$$

5. Finding elements of non - key elements.

New element

= Existing element - (FR x corresponding element in the key row)

Elements of First Row :

= 125 - (5/27 x 135) = 100

= 1 - (5/27 x 0) = 1

$$= \frac{1}{2} - \left[\frac{5}{27} \times \frac{3}{2}\right] = \frac{1}{2} - \frac{5}{8} = \frac{9-5}{18} = \frac{2}{9}$$

$$= \frac{5}{8} - \left[\frac{5}{27} \times \frac{27}{8}\right] = 0 = \frac{1}{8} - \left[\frac{5}{27} \times -\frac{9}{8}\right]$$

$$\frac{1}{8} + \frac{5}{24} = \frac{3+5}{24} = \frac{1}{3}$$

$$= 0 - \left[\frac{5}{27} \times 0\right] = 0 = 0 - \left[\frac{5}{27} \times 1\right] = -\frac{5}{27}$$

$$= 0 - \left[\frac{5}{27} \times 0\right] = 0$$

$$= 0 - \left[\frac{5}{27} \times 1\right] = -\frac{5}{27}$$

Element of Second Row

= 25 - (-1/27 x 135) = 25 +5 = 30

$$= 0 - \left[\frac{-1}{27} \times 0\right] = 0$$

$$= \frac{1}{2} - \left[\frac{-1}{27} \times \frac{3}{2}\right] = \frac{1}{2} + \frac{1}{18} = \frac{9+1}{18} = \frac{5}{9}$$

$$-\frac{1}{8} - \left[\frac{-1}{27} \times \frac{27}{8}\right] = \frac{-1}{8} + \frac{1}{8} = 0$$

$$= \frac{-5}{8} - \left[\frac{-1}{27} \times \frac{-9}{8}\right] = \frac{-5}{8} - \frac{1}{24} - \frac{1}{8} = \frac{-15-1}{24} = \frac{16}{24} = \frac{-2}{3}$$

$$= 0 - \left[\frac{-1}{27} \times 1\right] = \frac{1}{27}$$

6. NER of x_3 = 0 because it is a key column element

   NER of x_2 = NER Value of x_2 in Initial simplex table - (KC element of initial simplex Tablex x First value in x_2 column of Iteration 2 + NER value of x_3 column x third element of x_2 column of Iteration 2)

$$= 144 - \left(270 \times \frac{2}{9} + 225 + \frac{4}{9} \right) = -ve$$

$$\text{NER of } S_1 = 0 - \left(270 \times \frac{1}{3} + 225 \times -\frac{1}{3}\right) = -ve$$

$$\text{NER of } S_3 = 0 - \left(270 \times \frac{-5}{27} + 225 \times \frac{8}{27}\right) = -ve$$

7. Since all NER values are 0 and -ve, it has rached optiomal stage.

   Note : Key columnn (K.C) : It is column with max +Ve element (incease of maximisation LPP's)

$$\text{R.R of any row} = \frac{\text{Element in Qty column}}{\text{Element in key column}}$$

Key row (K.R) : It is the row with least +Ve,element.

Key Element (K.E) : It is the intersection element of Key column and key row

K.C : Note - The variables which lies along the Key column is called incoming variable.

K.R Note - The variables which lies along the Key row is called out going variable.

Example 2 :

A co. makes two kinds of leather belts. Belt A is a high quality belt & Belt B is of a low quality. The respective profity are Rs. 0.40 & Rs. 0.30 per belt. Such belt of type. A requires twice as much time as a belt of type B's if all the belts are of type B the co. can make 1000 day. The supply of leather is sufficient for only 800 belts per day [both A and B combined]. Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles per day available for Belt B(1) Formulate the problem as LPP (2) using simplex method find the optimum daily production of each type of belt. What is the maximum daily profit ?

Solution : LP formulation

Let x_1 x_2 numbers of belt A and belt B respectively daily.

Z = Profit

Max $Z = 0.4\ x_1 + 0.30\ x_2$

Subject to :

1. $2x\dfrac{1}{1000}x_1 + \dfrac{1}{1000}\ x_2 \leq 1$ (Constraint on time)

   $2x_1 + x_2 \leq 1000$ (Constraint on time)
2. $x_1 + x_2 \leq 800$ (Constraint on leather)
3. $x_1 \leq 400$ (Constraint on fancy buckle of belt A)
4. $x_2 \leq 700$ (Constraint on buckle of belt B)

   $(x_1 , x_2 \geq 0)$

Introducing slack variables S_1, S_2 ,S_3 , S_4 in constrainst 1,2, 3 & 4 respecitvely. The above LPP is modified as

Max Z = $Z = 0.4\ x_1 + 0.30\ x_2 + OS_1 + OS_2 + OS_3 + OS_4$

1. $2x_1 + x_2 + S_1 = 1000$
2. $x_1 + x_2 + S_2 = 800$
3. $x_1 + S_3 = 400$
3. $x_2 + S_4 = 700$

   $(x_1 , x_2 ,\ S_1, S_2 , S_3 , S_4 \geq 0)$

Number of variables = 6,

No. of equation = 4.

$\therefore$ 6 - 4 = 2 variables

i.e., x_1 = 0 x_2 = 0

∴ S_1= 1000 ; S_2 = 800 , S_3 = 400 and S_4 = 700

Initial Simplex table

FR	Prog	Pft P.U.	Qty	0.40 x_1	0.30 x_2	0 S_1	0 S_2	0 S_3	0 S_4	RR
2	S_1	0	1000	2	1	1	0	0	0	1000/2 = 500
1	S_2	0	800	1	1	0	1	0	0	700/1 =700
-	S_3	0	400	1 K.E	0	0	0	1	1	400/1 KR =400
0	S_4	0	100	0	1	0	0	0	1	700/0 = ∞
	NER			0.40 K.C	0.30	0	0	0	0	

Iteration 1

FR	Prog	Pft P.U.	Qty	x_1	x_2	S_1	S_2	S_3	S_4	RR
-	x_1	0	200	1	1 K.E	1	0	-2	0	200/1 = 200K.R
1	S_2	0	400	0	1	0	1	-1	0	200/1=400
0	S_3	0.40	400	0	0	0	0	1	1	400/0 = ∞
1	S_4	0	700	0	1	0	0	0	1	700/1 =700
	NER			0	0.30 K.C.	0	0	-ve	0	

Iteration 2

FR	Prog	Pft P.U.	Qty	x_1	x_2	S_1	S_2	S_3	S_4	RR
-2	x_2	0.30	200	0	1	1	0	-2	0	-ve
-	S_2	0	200	0	0	-1	1	1 KE	0	200/1 K.R= 200
1	x_1	0.40	400	1	0	0	0	1	0	400/1=400
2	S_4	0	500	0	0	-1	0	2	1	500/2=250
	NER			0	0	-ve	0	0.20 K.C	0	

Iteration 3

FR	Prog	Pft P.U.	Qty	x_1	x_2	S_1	S_2	S_3	S_4	RR
	x_2	0.30	600	0	1	-1	2	0	0	
	S_3	0	200	0	0	-1	1	1	1	
	x_1	0.40	200	1	0	1	-1	0	0	
	S_4	0	100	0	0	-1	-2	0	0	
	NER			0	0	-ve	-ve	0	0	[Optimal state]

Hence the optimal solution is given by

$x_1 = 200$ $x_2 = 600$

Max Z = [0.40 x 200 + 0.30 x 600] = Rs. 260

Hence the company must make 200 numbers of belts A and 600 nos. of Belt B daily in order to derive the maximum profit of Rs. 260, per day

Iteration 1

Now element = Existing element - FR x corresponding element in the Key row

= 1000 - [2 x400] = 200

= 1 - [2 x0] = 1

= 0 - [2 x1] = -2

= 800 - [1 x 400] = 400

= 1- [1x 0] = 1

= 1- [1x 1] = -1

NER of x_2 = 0.30 - [0.40 x 0] = 0.30

NER of S_3 = 0 - [0.40 x 1] = -ve

Iteration 2

= 400 - [1 x 200] = 200

= 0 - [1 x 1] = -1

= -1 - [1 x -2] = 1

= 700 - [1 x 200] = 500

= 0- [1x 1] = -1

= 0 - [1x -2] = 2

NER of S_1 = 0 - [0.30 x 1 + 0.40 x 0] = -ve

NER of S_3 = 0 - [0.30 x 2 + 0.40 x 0] = 0.20

Iteration 3

= 200 - [-2 x 200] = 600

= 1 - [-2 x 1] = -1 = 0 - [-2 x 1] = 2

= 400 - [1 x 200] = 200 = 0 - [1x -1] = 1

= 500 - [2 x 200] = 100

= 1- [2x -1] = 1 = 0 - [2x 1] = -2

NER of S_1 = 0 - [0.30 x -1 + 0.40 x 0] = -ve

NER of S_2 = 0 - [0.30 x 2 + 0.40 x -1] = -ve

In Iteration 3, NER values are Zero and negative there fore is has reached optimal stage.

Example 3 :

Paint bruch ltd. produces both exterior and interior paints from two raw materials R_1 and R_2. Following tables provides the relevant data.

Particulars	Tons of raw material pertons		Maximum daily avaiability (in tons)
	Exterior Paint	**Interior Paint**	
Raw material R_1	6	4	24
Raw material R_2	1	2	6
Profit Per ton (Rs. '000)	5	4	

Market survey shows, that not more than 2tons of interior point can be sold per day. You are required to determine the optimum product mix of interior and exterior paints to maximise the total daily profit.

Solution : LP formulation

Let x_1 tons and x_2 tons of exterior & interior paint be respectively produced daily.

Max $Z = 5000\ x_1 + 4000\ x_2$

Subject to :

1. $6\ x_1 + 4\ x_2 \leq 24$ (Constraint on Raw material R_1)
2. $1\ x_1 + 2\ x_2 \leq 6$ (Constraint on Raw material R_2)
3. $x_2 \leq 2$ (Constraint on sale of interior paint)

$(x_1 , x_2 \geq 0)$

Introducing slack variables S_1, S_2 and S_3 in constrainst 1,2, and 3 respectively The LPP is modified as

Max $Z = 5000\ x_1 + 4000\ x_2 + OS_1 + OS_2 + OS_3$

1. $6\ x_1 + 4\ x_2 + S_1 = 24$
2. $1\ x_1 + 2\ x_2 + S_2 = 650$
3. $x_2 + S_3 = 2$

$(x_1 , x_2 , S_1, S_2 , S_3 \geq 0)$

Number of variables = 5,

No. of equation = 3.

$\therefore$ 5 - 3 = 2 variables are equal to 0.

$x_1 = 0$ $x_2 = 0$ $S_1 = 24$;

$\therefore S_2 = 6$ and $S_3 = 2$

Intial Simplex table

FR	**Prog**	**Pft P.U.**	**Qty**	**5000 x_1**	**4000 x_2**	**0 S_1**	**0 S_2**	**0 S_3**	**RR**
	S_1	0	24	6 K.E	4	1	0	0	24/6 = 4 (KR)
1/6	S_2	0	6	1	2	0	1	0	6/1 = 6
0	S_3	0	2	0	0	0	0	1	2/0 = ∞
N-ER	5000 KC	4000	0	0	0	0			

Iteration 1

FR	**Prog**	**Pft P.U.**	**Qty**	**x_1**	**x_2**	**S_1**	**S_2**	**S_3**	**RR**
1/2	x_1	5000	4	1	2/3	1/6	0	0	4x3/2 = 6
-	S_2	0	2	0	4/3 KE	-1/6	1	0	2x3 /4 = 3/2 (KR)
3/4	S_3	0	2	0	1	0	0	1	2/1 = 2
NER				0	2000/3 KC	-ve	0	0	

Iteration 2

FR	Prog	Pft P.U.	Qty	x_1	x_2	S_1	S_2	S_3	RR
	x_1	5000	3	1	0	1/4	-1/2	0	
	x_2	4000	3/2	0	1	-1/8	3/4	1	
	S_3	0	1/2	0	0	1/8	-3/4	1	
	NER			0	0	-ve	-ve	0	[Optimal state]

The optimal solution

$x_1 = 3 \;\; x_2 = 3/2$

$\therefore$ Max Z = 5000 x 3 + 4000 x 3/2= Rs. 21,000

Hence the co, must produce 3 tons of exterior paint and 3/2 tons of interior point in order to achieve the maximum profit of Rs. 21000 per day.

Calculations : Iteration 1

1. Divide the first row by 6 we get for $x_1 \; \frac{6}{6} = 1$, $x_2 = \frac{4}{6} = \frac{2}{3}$,$S_1 = \frac{1}{6}$, $S_2 = \frac{0}{6} = 0$,

   $S_3 = \frac{0}{0} = 0$

2. Finding elements of non - key elements

   New element

   = Exiting element - (FR x corresponding element in the key row)

   For second row elements

   = 6- [1/6 x 24] = 2   = 1 - (1/6 x 6) = 0

   = 2 - (1/6 x 4) = 4/3   = 0 - (1/6 x 1) = -1/6

   = 1 - (1/6 x 0) = 1   = 0 - (1/6 x 0) = 0

For third row elements

= 2- [0 x 24] = 2 = 2 - (0 x 6) = 0

= 1 - (0 x 4) = 1 = 0 - (0 x 1) = 0

= 0 - (0 x 0) = 0 = 0 - (0 x 0) = 0

3. Ner of x_1 = 0

   NER of x_2 = 4000 - (5000 x 2/3) = 2000/3

   NER of S_1 = 0 - (5000 x 1/0) = -ve

4. Since positive value exist optimal stage is not rached

5. Find KC, KE, RR, FR

Iteration 2

1. Introduce x_2 and drop S_2
2. Divide KE row (i.e second row) by 4/3
3. Find elements of Non - Key elements.

   First row elements

   Q+y = 4 - [1/2 x 2] = 3

   S_1 = 1/6 - (1/2 x -1/6) = 1/6 + 1/12 = 1/4

   S_2 = 0 - (1/2 x 1) = 1

   Third row elements

   Q+y = 2 - [3/4 x 2] = 1/2

   S_1 = 0 - (3/4 x -1/6) = 1/8

   S_2 = 0 - (3/4 x 1) = -3/4

4. NER of S_1 = Existing element of Initial simple table of S_1

   - [NER element of initial simplex Table of x_1 X element in S_1 column of x_1 row Iteration 2 + NER element simplex table of x_2 X element in S_1 column of r_2 row of Iteration 2)

   NER of S_1 = 0 - $\left(5000 \text{ x } \frac{1}{4} + 4000 \text{ x } -\frac{1}{8}\right)$ = -ve

   NER of S_2 = 0 - $\left(5000 \text{ x } \frac{-1}{2} + 4000 \text{ x } \frac{3}{4}\right)$ = -ve

5. Since the NER values are 0 and - ve it has reached optimal stage.

20.4 MINIMIZATION LPP'S [Using simplex mehtod] :

Example 4 :

Min $Z = 3x_1 + 4x_2$

Subject to :

1. $7x_1 + 9x_2 \leq 10$
2. $5x_1 + 4x_2 \geq 12$
3. $6x_1 + 7x_2 \geq 15$

   $x_1, x_2 \geq 0$

Surplus variables : To convert a constrain inequation of the type " $\geq$ " into an equation,we subtract a positive quanitity from the left hand side of the constrain. It is imaginary variable, called surpuls variable. The co- efficient in the objective function of the surplus variables is taken as zero.

In the above ' LPP ' we introduced slack variable S_1 in constrain one and surplus variables S_2 and S_3 in constrains two & three respectively.

Hence the LPP is modified as

Min $Z = 3x_1 + 4x_2 + OS_1 + OS_2 + OS_3$

1. $7x_1 + 9x_2 + S_1 = 10$
2. $5x_1 + 4x_2 + S_2 = 12$
3. $6x_1 + 7x_2 + S_3 = 15$

   $(x_1, x_2, S_1, S_2, S_3 \geq 0)$

Number of variables = 5,

No. of equation = 3.

$\therefore$ 5 - 3 = 2 variables are equal to 0.

i.e., $x_1 = 0$ $x_2 = 0$

$\therefore S_1 = 10$; $-S_2 = 12 \Rightarrow S_2 = -12$ and $-S_3 = 15 \Rightarrow S_3 = -15$

Note : In the above, negative values of the surplus variables S_2 and S_3 contradict the assumption that all the LP variables (real or imaginary should be non - negative).

Hence to overcome this situation. We introduce another imaginary variable called artifical variable to accompany each surplus variable. The co-efficient in the objective function of each artifical variable is taken as +M (i.e + ∞) in case of minimization LPP's. Hence by introduing arifical variables A_1 and A_2 in constrain two and three respectively the above LPP is further modified as -

Min Z = $3 x_1 + 4 x_2 + OS_1 + OS_2 + OS_3 + MA_1 + MA_2$

Subject to :

1. $7 x_1 + 9 x_2 + S_1 = 10$
2. $5 x_1 + 4 x_2 + S_2 + A_1 = 12$
3. $6 x_1 + 7 x_2 + S_3 + A_2 = 15$

   $(x_1 , x_2 , S_1, S_2 , S_3 , A_1 , A_2 \geq 0)$

Number of variables = 7,

No. of equation = 3.

$\therefore$ 7 - 3 = 4 variables are equal to 0.

i.e., $x_1 = 0$ $x_2 = 0$, $S_2 = 0$, $S_3 = 0$

$\therefore S_1 = 10$; $A_1 = 12$; $A_2 = 15$

Example 5 :

A firm plans to purchase at lest 200 kgs. of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain atleast 100 kgs of metal X and no more than 35 Kgs of metal Y. The firm can purchase the scrap from two suppliers [A and B] in unlimited quantities. The percentage of X and Y metals in terms of weight of the scraps supplied by A and B is given below.

Metal	Supplier A	Supplier - B
X	25%	75%
Y	10%	20%

The price of A s scrap is Rs. 200 per kg and that of 'B' s Rs. 400 per Kg. The firm wants to determine the quantities that it should buy form the two suppliers so that the total cost is minimised. Use simplex method and solve the problem.

Solution : LP formulation

Let the firm purchases a Kgs and b kgs of scrap from the two supplies A and B respectively.

Min Z = 200 a + 400 b

Subject to :

1. $a+b \geq 200$ (Constraint on total weight of scrap)
2. $25 a + 75 b \geq 100$ (Constraint on i.e. $1/4a + 3/4b \geq 100$ multiply the equation by $4a+3b \geq 400$)

3. 10/100a + 20/100b $\leq$ 35 (Constraint on metal y)

1/10a + 2/10b $\leq$ 35 multiply by 10.

a + 2 b $\leq$ 350

Introducing surplus variable S_1 and artifical variable A_1 in constraint 1, surplus variable S_2 and artificial variable A_2 in constraint 2, Slack variable S3 in constrain 3. The LPP is modified as:

Min Z = 200a + 400 b + OS_1 + OS_2 + OS_3 + MA_1 + MA_2

Subject to :

1. a + b - $S_1 A_1$ = 200
2. a + 3b - S_2 + A_2 = 400
3. a + 2b + S_3 + A_2 = 250

(a,b, S_1, S_2 , S_3 , A_1 , A_2 $\geq$ 0)

Number of variables = 7,

No. of equation = 3.

$\therefore$ 7 - 3 = 4 variables are equal to zero.

i.e., a =0 ; b = 0 , S_1 = 0, S_2 = 0

$\therefore$ A_1= 400 ; A_2 = 400 ; S_3 = 350

Simplex Table

FR	**Prog**	**Cost P.U.**	**Qty**	**200 a**	**400 b**	**0 S_1**	**0 S_2**	**0 S_3**	**+M A_1**	**+M A_2**	**RR**
1/3	A_1	+M	200	1	1	-1	0	0	1	0	200
-	A_2	+M	400	1	3 K.E.	0	-1	0	0	1	400/3 K.R.
2/3	S_3	0	350	1	2	0	0	1	0	0	350/2
	NER			200.2 m	400- 4m KC	+	+	0	0	0	

Iteration 1

FR	Prog	Cost P.U.	Qty	200 a	400 b	0 S_1	0 S_2	0 S_3	+M A_1	+M A_2	RR
-	A_1	+M	200/3	2/3 K.E	0	-1	1/3	0	1	-	100 KR
1/2	b	400	400/3	1/3	1	0	-1/3	0	0	-	400
1/2	S_3	0	250/3	1/3	0	0	2/3	1	0	-	250
	NER			-2M/3 +200/3 KC.	0	+	-M/3 + 400/3	0	0		

Iteration 2

FR	Prog	Cost P.U.	Qty	200 a	400 b	0 S_1	0 S_2	0 S_3	+M A_1	+M A_2	RR
-	a	200	100	1	0	-3/2	1/2	0	-	-	-
-	b	400	100	0	1	1/2	-1/2	0	-	-	-
-	S_3	0	50	0	0	1/2	1/2	1	-	-	-
	NER			0	0	+	+	0	0		

$\therefore$ The optimal solution is given by

a = 100 , b = 100

Min Z = 200 [100] + 400 [100] = Rs. 60,000

Hence the firm should purchase 100 kgs of scrap from each of the two suppliers. A and B in order to minimise the total purchase cost which is Rs. 60,000.

Steps for calculation :

1. In the first table in programme column enter A_1, A_2 and S_3.

2. In cost per unit column enter coefficients of A_1, A_2 and S_3 i.e +M, +M and 0).

3. In quantity column enter right hand side values of constraints equations.
4. In ' a column enter coefficients of 'a' of all the three constraints.
5. In 'b' column enter coefficients of 'b' of all the three constraints.
6. In ' s' column enter unit matrix with -ve sign.
7. In 'A' Column enter +ve unit matrix
8. Now find NER

   NER of = coefficient of 'a' in objective function - ['a' column a elements x cost per unit column elements]

   NER of a = 200 - [1M + 1M x 1(0)] = 200 - 2 M

   NER of b = 400 - [1M + 3M + 0(0)] = 400 - 4 M

   NER of S_1= 0 - [- 1M + M(0) + 0(0)] = + M

   NER of S_2= 0 - [M (0)+ -1 M + 0(0)] = + M

   NER of S_3= 0 - [(0) M + M(0) + 0(1)] = 0

   NER of A_1= M - [1M + 0(M) + 0(0)] = 0

   NER of A_2= M- [0 (M) + 1(M) + 0(0)] =0
9. Now take the least -ve value as KC, i.e, among 200 - 2M, 400 - 4 M take -2M and -4M. Here the least -e valus is - 4M.

   $\therefore$ 400 - 4 M is KC.
10. Find RR column i.e $\dfrac{\text{Quantity}}{\text{Key column Element}}$

    $\dfrac{200}{1} = 200, \dfrac{400}{3}$ and $\dfrac{350}{2}$ = 175 consider the least value i.e, KR.
11. The intersection value of KC and KR is KE i.e, 3
12. Find FR column i.e, $\dfrac{\text{Key column Element}}{\text{Key element}}$ i.e $\dfrac{1}{3}, \dfrac{2}{3}$

Second Table :

1. In programme column introduce 'b' and drop 'A_2'
2. Remove 'A_2' column
3. Convert K E to '1' Divide the row by 3.
4. Element of Non - Key Row = element of first table - [FR x corresponding element in KR]

   200 - [1/3 x 400] = 200/3

   1- [1/3 x 1] = 2/3

   -1- [1/3 x 0] = -1

   0- [1/3 x -1] = 1/3

   350 - [2/3 x 400] = 250/3

   -1- [2/3 x 1] = 1/3

   0- [2/3 x 0] = 0

   0- [2/3 x -1] = 2/3
5. NER of a = 200 - [M x 2/3 + 400 x 1/3] = -2M/3 + 200/3

   NER of S_1= 0 - [M x -1] = +

   NER of S_2= 0 - [M x 1/3 + 400 x -1/3] = M/3 + 400/3
6. Since in NER row there is -ve value, it has not reached optimal stage.
7. Repeat same procedure.
8. Find KC as least -ve value i.e. -2M/3 + 200/3
9. Find RR = $\frac{\text{Quantity}}{\text{Key column Element}}$
10. Select the least value i.e, KR.
11. The intersection value of KC and KR is KE
12. Find FR

Third Table :

1. In programme column introduce 'a' and drop 'A_1'
2. Remove 'A_1' column
3. Element of Non - Key row.

Second Row

Qty = 400/3 - [1/2 x 200/3] = 100

a = 1/3 - [1/2 x 2/3] = 0

b = 1 - [1/2 x 0] = 1

S_1 = 0 - [1/2 x -1] = 1/2

S_2 = -1/3 - [1/2 x 1/3] = -1/2

Third Row

Qty = 250/3 - [1/2 x 200/3] = 50

a = 1/3 - [1/2 x 2/3] = 0

S_1 = 0 - [1/2 x -1] = 1/2

S_2 = 2/3 - [1/2 x 1/3] = 1/2

4. NER of S_1= 0 - [200 x 3/2 + 400 X 1/2] = +100

   NER of S_2= 0 - [200 x 1/2 + 400 - 1/2] = +100

5. Since NER value are 0 and +ve it has reached optimal stage.

Example 6 :

A company sells two types of fertilizers, one is liquid and the other is dry. The liquid fertilizers contains two units of chemical A and 4 units of chemical B per jar and the dry fertilizer contains 3 units of each of the chemicals A & B per carton. The liquid fertilizer sells for Rs. 3 per jar and the dry fertilizer sells for Rs. 4 per carton. A farmer requires at least 90 units of chemical A and at least 120 units of chemical B for his farm. How many of each type of fertilizers should the farmer purchase to minimise the cost while meeting his requirement ? Use simplex method to solve the probelm.

Solution : LP formulation

Let the farmer purchase x_1 jars of liquid fertilizer and x_2 cartons of dry fertilizer for his farm.

Max Z = $3x_1 + 4x_2$

Subject to :

1. $2x_1 + 3x_2 \geq 90$ (Constraint on chem. A)
2. $4x_1 + 3x_2 \geq 120$ (Constraint on chem. B)

   (x_1 , $x_2 \geq 0$)

Introducing surplus variables S_1 and artificial variable A_1 in constrain one ; Surplus varaible S_2 and artifical variable A_2 in constrain two.

The LPP is modified as

Min Z = 3 x_1 + 4 x_2 + OS_1 + OS_2 + M A_1 + MA_2

1. 2 x_1 + 3 x_2 - S_1 + A_1 = 90
2. 4 x_1 + 3 x_2 - S_2 + A_2 = 120

(x_1 , x_2 , x_3 , S_1, S_2 , A_1, $A_2 \geq 0$)

Number of variables = 6,

No. of equation = 2.

$\therefore$ 6-2 = 4 variables are equal to zero

i.e., $x_1 = 0$ $x_2 = 0$ and $S_1 = 0$; $S_2 = 0$

$\therefore$ $A_1 = 90$; $A_2 = 120$

Simplex Table

FR	**Prog**	**Cost P.U.**	**Qty**	**3 x_1**	**4 x_2**	**0 S_1**	**0 S_2**	**+M A_1**	**+M A_2**	**RR**
1/2	A_1	+M	90	2	3	-1	0	1	0	90/2=45
-	A_2	+M	120	4 K.E	3	0	-1	0	1	120/4=30 K.R.
	NER			3.6 m	4.6 M	+	+	0	0	

Iteration 1

FR	**Prog**	**Cost P.U.**	**Qty**	**3 x_1**	**4 x_2**	**0 S_1**	**0 S_2**	**+M A_1**	**+M A_2**	**RR**
-	A_1	+M	30	0	3/2	-1	1/2	1	0	30/3/2=20
1/2	x_1	3	30	1	3/4	0	-1/4	0	1	30/3/4=40
	NER			0	-3 M/2 +7/4	+	-M/2 +3/4	0	0	

Iteration 2

FR	Prog	Cost P.U.	Qty	3 x_1	4 x_2	0 S_1	0 S_2	+M A_1	+M A_2	RR
-	x_2	4	20	0	1	-2/3	1/3	1	0	
-	x_1	3	15	1	0	1/2	-1/2	0	1	
	NER			0	0	+	+			[Optimal stage]

Hence the optimal solution is given by

$x_1 = 15$ $x_2 = 20$

Max Z = 3 x 15 + 4 x 20 = Rs. 125

Hence the farmer must purchase 15 units of liquid fertilizer and 20 units of dry fertilizer in order to minimise the cost.

Calculations :

First Table

NER of x_1 = 3 - (2m + 4m) = 3-6m

NER of x_2 = 4 - (3m+3m) = 4-6m

NER of S_1 = 0 - (-m) = +m

NER of S_2 = 0 - (0 + Mx - 1) = +

Second Table :

3rd row

90- [1/2 x120] = 30 ,

3 - [1/2 x 3] = 3/2

-1[-1/2 x0] = -1

0-[1/2 x -1] = 1/2

NER of x_2 = 4 - (3/2 m+3 x 3/4) = -3m/2 + 7/4

NER of S_1 = 0 - (M x -1 + 3 x 0) = +

NER of S_2 = 0 - (M+ 1/2 + 3 x -1/4) = -M/2 + 3/4

Third Table :

30- [1/2 x 30] = 15 ,

0 - [1/2 x -1] = 1/2

-1/4 [1/2 x 1/2] = -1/2

NER of S_1 = 0 - (4 x -2/3 + 3 x 1/2) = +

NER of S_2 = 0 - (3+ -1/2 + 4 x 1/3) = +

20.5 SUMMARY :

When the decision variables are three or more the graphical method becomes impossible in such cases, this method is very useful. It is an algebraic method which can deal with any number of decision variables.

20.6 QUESTIONS & EXERCISES :

1. Explain the simplex method of solving LP problem.

2. A farm is engaged in bedding Pigs. The Pigs are fed on various products grown on the farm. Because of the need to ensure certain nutrient constituents, it is necessary to buy additional one or two products which we shall call as A and B. The nurtrient constituent (vitamins & proteins) in each unit of the product are given below)

Nutrient Constituent	Nubient contentsin the product		Minimum Requirement of nutrient constituent
	A	**B**	
1	36	6	108
2	3	12	36
3	20	10	100

Product A costs Rs. 20 P.U. and product B costs Rs. 40 Determine how much of product A & B must be purchased so as to provide the pigs nutrients not less than the minimum required at the lowest possible cost. Solve the probelm using simplex method.

[Ans : Min Z = 20 a + 40 b

Subject to :

$36a + 6b \geq 108$

$3a + 12b \geq 36$

$20a + 10b \geq 100$

{a, b ≥ 0}]

3. An animal feed co. must produce 200 kgs of a mixture consisting of ingerdients x_1 and x_2 daily, x_1 cost Rs. 3 per Kg and x_2 per 8 per kg. No more than 80 kgs of x_1 can be used and at least 60 kgs of x_2 must be used. Find how much of each ingredient should be used if the co. wants to minimise the cost ?

[Ans : Min $Z = 3x_1 + 8x_2$

Subject to :

$x_1 + x_2 = 200$

$x_1 \leq 80$

$x_2 \geq 60$

$x_1, x_2 \geq 0$]

4. Solve the following linear programming problems by simplex method.

(a) Max $Z = 3x_1 + 2x_2$

Subject to :

$-2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1 + x_2 \leq 3$

$x_1 + x_2 \geq 0$

(b) Max $Z = 10x_1 + 6x_2$

Subject to :

$x_1 + x_2 \leq 2$

$2x_1 + x_2 \leq 4$

$3x_1 + 8x_2 \leq 12$

$x_1, x_2 \geq 0$

(c) Max $Z = 50x_1 + 60x_2$

Subject to :

$2x_1 + 3x_2 \leq 1500$

$3x_1 + 2x_2 \leq 1500$

$x_1 \leq 450$

$x_1, x_2 \geq 0$

[Ans : x_1 =300 , x_2 = 300 with Z = 50x300+60x300 = 33,000]

5. A manufacturer makes two products P_1 and P_2 using two machines M_1 and M_2. Product P_1 requires 5 hours on machine M_1 and no time on machine M_2. There are 16 hours of time per day available on machine M_1 and 30 hours on M_2. Profit margin from P_1 and P_2 is Rs. 200 and Rs. 10.00 per unit respectively. What should be the daily production mix to optimize profit ?

6. A manufacturer produces two items x_1 and x_2 . x_1 needs 2hours on machine A and 2 hours on machine B. x_2 needs 3 hours on machine A and 1 hour on machine B. If machine A can run for a maximum of 12 hour per day and B for 8 hours per day and profits from x_1 and x_2 are Rs. 4 and 5 per item respectively, find by simplex method, how many items per day be produced to have maximum profit. Give the interpretation for the values of indicators corresponding to slack variables in the final iteration.

[Ans : 3. 2., Rs 22]

7. A producer produces three commodities x,y and z and has two limited inputs storages and petrol. To produce a unit either of x or y 2 units of storage and 1 unit of petrol are required while a unit of z requires only two units of storage and no petrol. The producer has only 8 units of storage space and 3 units of petrol. If each unit of x,y and z gives Rs.3, Rs. 7 and Rs. 6 respectviley of profits how much of each should the producer produce to maximize his profit ?

[Ans : 0.3.1., Rs 27]

8. Anita eletric company produces two products P_1 and P_2 products are produced and sold on a weekly basis. The weekly production cannot exceed 25 for product P_1 and 35 for product P_2 because of limited available facilities. The company employs a total of 60 workers. Product, P_1 requiers 2 man - weeks labour. is while P_2 requiers one man - week labour. Profit margin on P_1 is Rs. 60 and one P_2 is Rs. 40. Formulate it a linear programming problem and solve for maximum profit.

[Ans : 12.5, 35 and Max Profit = 1.475]

9. (a) Minimize (cost) $Z = 2y_1 + 3y_2$

Subject to the constraints :

$y_1 + y_2 \geq 5$

$y_1 + 2y_2 \geq 6$

$y_1 \geq 0$ and $y_2 \geq 0$

Determine the minimum cost

[Ans : $y_1 = 4$, $y_2 = 1$ and minimum cost is 11]

(b) Minimize $Z = 4 x_1 + 2 x_2$

Subject to the constraints

$3x_1 + x_2 \geq 27$

$-x_1 - x_2 \geq 21$

$x_1 + 2x_2 \geq 30$

$x_1 , x_2 \geq 0$

(c) Minimize $Z = 3 x_1 + 2.5 x_2$

Subject to the constraints

$2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1 \geq 0 , x_2 \geq 0$

[Ans : $x_1 = 15$, $x_2 = 5/2$]

(b) Minimize $Z = 3 x_1 + 4 x_2$

Subject to the constraints

$4x_1 + x_2 \geq 300$

$-x_1 - x_2 \leq -18$

$x_1 + 3 x_2 \geq 28$

$x_1 , x_2 \geq 0$

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